Global and Stochastic Analysis, Vol.9 No. 2 (March, 2022)
Special Issue: Modern Problems of Equations of Mathematical Physics and its Applications

SOLVING THE HEAT EQUATION BY SOLVING AN INTEGRO-DIFFERENTIAL EQUATION

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Abstract. In this article we build a solution to the well-known problem of heat propagation from a point source. The solution was achieved through a partially unusual and approximate approach. At a certain stage of this development, we arrived at a linear and inhomogeneous Volterra-type integro-differential equation for a real function, which is connected with the solution of the heat equation by a specific integral transformation over the Fourier transform of the increment function of temperature.

1. Introduction

Heat propagation is one of the physical processes that is standardly addressed in traditional mathematical physics textbooks [1], [2] and in books on partial differential equations [3], [4]. The dynamics of this propagation is contained in an equation that has been solved for different situations, conditions and dimensions. From the point of view of the mathematical methods used in the developments linked to this process, we have, for example, that within the context of integral equations [5], [6], the solution of the equation for heat propagation can be expressed, precisely, by a specific integral equation [7]. In this paper, we present an approximate solution for the heat propagation equation, generated by a fixed-point source inside a material medium, which makes use of an adequate integral transformation, which, in turn, allows the emergence of an integro-differential equation (Volterra, linear and inhomogeneous) for a real function, an equation that was solved by the series method.

1.1. The heat equation. The problem presented here is to determine the function θ, of variables x, y, z, t (or, equivalently, $\vec{r}, t$), which represents the temperature increase of a linear, homogeneous and isotropic physical medium that occupies the half space $z \leq 0$ surrounding a point source of heat, located in the position $(0, 0, z_0)$ of the reference frame considered, for $t > 0$,

$$k\nabla^2\theta(\vec{r}, t) + F(\vec{r}, t) = m\frac{\partial}{\partial t}\theta(\vec{r}, t) \tag{1.1}$$

the term $F(\vec{r}, t)$ corresponding to the heat source, the same as, in the specific case, is defined as,

$$F(\vec{r}, t) = \lim_{\epsilon \to 0} \frac{Q(t)}{\epsilon^3},$$

1991 *Mathematics Subject Classification.* Primary 45D05; Secondary 35K05.
Key words and phrases. Heat equation, Integro-differential equation, Integral transform.

so that $F(\vec{r},t)d\tau$ represents the amount of heat generated per unit of time in the volume element $d\tau$ around point $\vec{r}$, and $Q(t)$ is the power of the source (amount of heat generated per unit of time).

2. Development

Let's transform the heat-source equation. The changing functions (and variables) as a result of the transformations to be used, as well as the corresponding notation, is summarized below:

$$\theta(t,x,y,z) \quad\longrightarrow\quad \bar{\theta}(s,x,y,z) \quad\longrightarrow\quad \bar{\Theta}(s,\alpha,\beta,z), \tag{2.1}$$

In equation (1.1) above, we will successively apply (i) the Laplace transform (in relationship to the time variable), which generates the expression,

$$m\int_0^\infty \frac{\partial}{\partial t}\theta(\vec{r},s)e^{-st}dt \;=\; K\Big(\bar{\theta}_{xx} \;+\; \bar{\theta}_{yy} \;+\; \bar{\theta}_{zz}\Big) \;+\; \bar{F}(\vec{r},t),$$

or,

$$ms\bar{\theta} \;=\; K\Big(\bar{\theta}_{xx} \;+\; \bar{\theta}_{yy} \;+\; \bar{\theta}_{zz}\Big) \;+\; \bar{F}(\vec{r},t), \tag{2.2}$$

where, according to the context of the problem, it was considered that $\theta(t=0)=0$, and also the double Fourier transform (in relationship to the variables x and y), generating the expression,

$$\begin{aligned}\Big(\frac{ms}{K}\Big)\bar{\Theta}(z,s,\vec{\alpha}) \;=\; &\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\theta}_{xx}\; e^{i(\alpha x\,+\,\beta y)}dxdy\; +\\ &+\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\theta}_{yy}\; e^{i(\alpha x\,+\,\beta y)}dxdy \;+\; \bar{\Theta}_{zz}\; +\\ &+\frac{1}{4\pi^2 K}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{F}(\vec{r},s)\; e^{i(\alpha x\,+\,\beta y)}dxdy.\end{aligned} \tag{2.3}$$

Note that we can write,

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\theta}_{xx}\; e^{i(\alpha x\,+\,\beta y)}dxdy = (-i\alpha)^2\bar{\Theta}(z,s,\vec{\alpha}), \tag{2.4}$$

where $\vec{\alpha} \;=\; (\alpha,\beta)$. Thus, the equation is generated,

$$\bar{\Theta}_{zz}(z,s,\vec{\alpha}) - \mu^2\bar{\Theta}(z,s,\vec{\alpha}) + \bar{f}(z,s,\vec{\alpha}) = 0, \tag{2.5}$$

where,

$$\mu^2 = \alpha^2+\beta^2+\frac{ms}{K} = \rho^2+\frac{ms}{K}. \tag{2.6}$$

Equation (2.5) written explicitly is,

$$\bar{\Theta}_{zz}(z,s,\vec{\alpha}) \,-\, \mu^2\bar{\Theta}(z,s,\vec{\alpha}) \,+\, \frac{1}{4\pi^2 K}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{F}(\vec{r},s)\; e^{i(\alpha x\,+\,\beta y)}dxdy \;=\; 0, \tag{2.7}$$

being,

$$\begin{aligned}&\bar{F}(\vec{r},s) = \lim_{\epsilon\to 0}\frac{\bar{Q}(s)}{\epsilon^3}, && |\vec{r}-z_0\hat{k}| < \epsilon,\\ &\bar{F}(\vec{r},s) = 0, && |\vec{r}-z_0\hat{k}| > 0,\end{aligned} \tag{2.8}$$

Note that the condition: $|\vec{r} - z_0\hat{k}| < \epsilon$, equivalent to writing,

$$|x| < \epsilon, \qquad \& \qquad |y| < \epsilon \qquad \& \qquad |z - z_0| < \epsilon,$$

The application of Fourier and Laplace transformations allowed the passage from an equation with three partial derivatives (relative to three independent variables) to an equation with only one partial derivative (relative to an independent variable). It is clear that the corresponding inverse transformations must then be applied.

Now let's look at the double integral in (2.7), which can be calculated directly,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{F}(\vec{r}, s)\, e^{i(\alpha x + \beta y)} dxdy \;=\; \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon} \frac{\bar{Q}(s)}{\epsilon^3}\, e^{i(\alpha x + \beta y)} dxdy \;=$$

$$= \lim_{\epsilon \to 0}\left\{ \frac{\bar{Q}(s)}{\epsilon} \int_{-\epsilon}^{\epsilon} \frac{e^{i\alpha x}}{\epsilon} dx \int_{-\epsilon}^{\epsilon} \frac{e^{i\beta y}}{\epsilon} dy \right\} \;=\; \lim_{\epsilon \to 0}\left\{ \frac{4\bar{Q}(s)}{\epsilon} \times \frac{sin(\alpha\epsilon)}{\alpha\epsilon} \times \frac{sin(\beta\epsilon)}{\beta\epsilon} \right\}$$

$$= 4 \lim_{\epsilon \to 0}\left(\frac{\bar{Q}(s)}{\epsilon} \right) \times \lim_{\epsilon \to 0}\left(\frac{sin(\alpha\epsilon)}{\alpha\epsilon} \right) \times \lim_{\epsilon \to 0}\left(\frac{sin(\beta\epsilon)}{\beta\epsilon} \right) = 4 \lim_{\epsilon \to 0}\left(\frac{\bar{Q}(s)}{\epsilon} \right). \quad (2.9)$$

So, we write,

$$\bar{f}(z, s) = \frac{1}{\pi^2 K} \lim_{\epsilon \to 0}\left(\frac{\bar{Q}(s)}{\epsilon} \right). \quad (2.10)$$

expression that could be interpreted as corresponding to a one-dimensional heat source.

2.1. Defining an integral transformation. We start by assuming that there is a real function G with two independent variables, whose integral will generate, when well defined, the transformed function $\bar{\theta}$. In mathematical terms, and just as a formal expression, we write the integral transformation,

$$\bar{\Theta}(z, s) = \int_{p(z)}^{q(z)} dz'\, G(z, z', s), \quad (2.11)$$

where $p(z)$ and $q(z)$ are values that, because they have been introduced in the problem, can be defined (chosen) properly.

On the other hand, using the equation for $\bar{\Theta}$, equation (2.2), one can deduce the corresponding equation for the function G. From (2.2) we see the need to determine the second partial derivative of expression (2.4) with respect to the variable z, which can be done using the 'second fundamental theorem of calculus', which states,

$$\frac{d}{dz}\int_{p(z)}^{q(z)} dz'\, G(z, z') \;=\; \int_{p(z)}^{q(z)} dz'\, \frac{\partial G}{\partial z}(z, z') \;+\; G(z, q)\frac{dq}{dz} \;-\; G(z, p)\frac{dp}{dz}, \quad (2.12)$$

where it results to,

$$\frac{d^2}{dz^2}\int_{p(z)}^{q(z)} dz'\, G(z, z') \;=\; \int_{p(z)}^{q(z)} dz'\, \frac{\partial^2 G}{\partial z^2}(z, z') \;+\; 2\frac{\partial G}{\partial z}(z, q)\frac{dq}{dz} \;+$$

$$-2\frac{\partial G}{\partial z}(z, p)\frac{dp}{dz} \;+\; G(z, q)\frac{d^2 q}{dz^2} \;-\; G(z, p)\frac{d^2 p}{dz^2}, \quad (2.13)$$

Looking at the expression above, we see that, to simplify it, it is convenient to define $q(z)$ and $p(z)$ equal to z and 0, respectively; in consequence, one can write,

$$\frac{\partial^2\Theta}{\partial z^2}(z,s) \;=\; \frac{d^2}{dz^2}\int_0^z dz'\, G(z,z') \;=\; \int_0^z dz'\, \frac{\partial^2 G}{\partial z^2}(z,z') + 2\frac{\partial G}{\partial z}(z,z'), \qquad (2.14)$$

expression that, after replacing it in equation (2.5), leads to the following,

$$\int_0^z dz' \left(\frac{\partial^2 G}{\partial z^2}(z,z') - \mu^2 G(z',z)\right) + 2\frac{\partial G}{\partial z}(z,z) + \bar{f}(z,s) \;=\; 0 \qquad (2.15)$$

In (2.15) we have a Volterra integro-differential equation, linear and inhomogeneous for the function G.

3. Solving the integro-differential equation

We show below that it is possible to construct a solution for equation (2.15) which is of the "series expansion" type. Let's suppose that, for certain coefficients $\varphi_n(s)$, for $n = 1, 2, 3, ...$, to be determined, and for a parameter[1] L, the value of the function G, i.e. $G(z, z', s)$, can be expressed by the following series,

$$G(z,z',s) = \sum_{n=1}^{\infty} \varphi_n(s)\, cos\Big(\frac{n\pi}{L}(z-z')\Big), \qquad (3.1)$$

from which it immediately follows that,

$$\frac{\partial G}{\partial z}(z, z' = z, s) = 0,$$

And, as a consequence of the above, equation (2.15) is simplified to the following,

$$\int_0^z dz' \left(\frac{\partial^2 G}{\partial z^2}(z,z') - \mu^2 G(z',z)\right) \;+\; \bar{f}(z,s) \;=\; 0. \qquad (3.2)$$

Note that the development in (3.1) does not correspond to a Fourier cosine series because from $G(z, z')$ the values of the coefficients $\varphi_n(s)$, will not be defined; which will define the value of $G(z, z')$.

Additionally, since the term $\bar{f}(z, s)$ is in principle known (as it is defined by the source), it is possible to express it in terms of a suitable Fourier sine series; so, we write,

$$\bar{f}(z,s) = \sum_{n=1}^{\infty} f_n(s)\, sin\Big(\frac{n\pi}{L}z\Big), \qquad (3.3)$$

in which the coefficients $f_n(s)$ are determined by the expression,

$$f_n(s) = \frac{2}{L}\int_0^L \bar{f}(z,s)\, sin\Big(\frac{n\pi}{L}z\Big)dz. \qquad (3.4)$$

[1]Corresponding to the minimum distance, measured from the source, where a θ remains equal to zero for $t > 0$.

To continue in accordance to (3.2), we need to calculate a second partial derivative and the integral of G. Through the right calculus we have,

$$\frac{\partial G}{\partial z}(z,z',s) \;=\; \sum_{n=1}^{\infty}\varphi_n(s)\Big(-\frac{n\pi}{L}\Big)\; sin\Big(\frac{n\pi}{L}(z-z')\Big), \tag{3.5}$$

$$\frac{\partial^2 G}{\partial z^2}(z,z',s) \;=\; \sum_{n=1}^{\infty}\varphi_n(s)\Big(-\frac{n^2\pi^2}{L^2}\Big)\; cos\Big(\frac{n\pi}{L}(z-z')\Big), \tag{3.6}$$

$$\int_0^z dz'\;\frac{\partial^2 G}{\partial z^2}(z,z',s) \;=\; -\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)\varphi_n(s)\; sin\Big(\frac{n\pi}{L}z\Big), \tag{3.7}$$

and,

$$\int_0^z dz'\; G(z',z) = \sum_{n=1}^{\infty}\left(\frac{L}{n\pi}\right)\varphi_n(s)\; sin\left(\frac{n\pi}{L}z\right). \tag{3.8}$$

Using the partial results above, expressions (3.3), (3.7) and (3.8), in (3.2), we obtain,

$$-\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)\varphi_n(s)\; sin\left(\frac{n\pi}{L}z\right) \;-\; \sum_{n=1}^{\infty}\left(\frac{\mu^2 L}{n\pi}\right)\varphi_n(s)\; sin\left(\frac{n\pi}{L}z\right) \;+$$

$$+\;\sum_{n=1}^{\infty} f_n(s)\; sin\left(\frac{n\pi}{L}z\right) \;=\; 0 \tag{3.9}$$

the same one that, in a compact way, can be rewritten as follows,

$$\sum_{n=1}^{\infty}\left(-\Big(\frac{n\pi}{L}+\frac{\mu^2 L}{n\pi}\Big)\varphi_n(s) \,+\, f_n(s)\right) sin\Big(\frac{n\pi}{L}z\Big) \;\equiv \sum_{n=1}^{\infty} a_n sin\Big(\frac{n\pi}{L}z\Big) \;=\; 0, \tag{3.10}$$

where a simplification was made in the representation of the coefficient within the sum that appears on the left side.

According to linear algebra, the functions $\{sin(n\pi z/L)\}$, for $n = 1, 2, 3, \ldots$, form an orthonormal (and therefore linearly independent) set in the interval $< 0, L >$; it follows from this that, in (3.10), $a_n = 0$, for $n = 1, 2, 3\ldots$, which, in turn, allows us to write,

$$a_n = 0 \quad\to\quad \varphi_n(s) = \left(\frac{n\pi}{L}+\frac{\mu^2 L}{n\pi}\right)^{-1} f_n(s). \tag{3.11}$$

In this way, the coefficients $\varphi_n(s)$ depend on the coefficients $f_n(s)$.

Next, we will calculate, precisely, the coefficients $f_n(s)$ using (2.10), (3.4) and what we have previously assumed: that the source is of the point type and is located in the position $(0, 0, z_0)$ of the reference frame considered. So, we wrote,

$$f_n(s) = \left(\frac{2}{\pi^2 KL}\right)\int_0^L \lim_{\epsilon\to 0}\left(\frac{\bar{Q}(s)}{\epsilon}\right) sin\Big(\frac{n\pi}{L}z\Big)dz,$$

$$f_n(s) = \left(\frac{2\bar{Q}(s)}{\pi^2 kL}\right)\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\int_{z_0-\epsilon}^{z_0+\epsilon} sin\Big(\frac{n\pi}{L}z\Big)dz\right). \tag{3.12}$$

where we have considered: $L > z_0$. After doing the integral in (3.12) we get,

$$f_n(s) = \left(\frac{2\bar{Q}(s)}{\pi^2 kL}\right)\left(\frac{L}{n\pi}\right)\left\{\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(cos\left(\frac{n\pi}{L}(z_0-\epsilon)\right)-cos\left(\frac{n\pi}{L}(z_0+\epsilon)\right)\right)\right\}, \quad (3.13)$$

Expression that can be rewritten using trigonometric identity; so, after arranging the terms, we have,

$$f_n(s) = \left(\frac{4\bar{Q}(s)}{\pi^2 kL}\right) sin\left(\frac{n\pi z_0}{L}\right)\left\{\lim_{\epsilon\to 0}\left(\frac{L}{n\pi\epsilon}\, sin\left(\frac{n\pi\epsilon}{L}\right)\right)\right\}, \quad n = 1, 2, 3... \quad (3.14)$$

Expression (3.14) is well defined for finite values of the index "n", but not for values $n \to \infty$. So, according to (3.11), we have that $\varphi_{(n\to\infty)}(s)$ is not defined either. Consequently, to avoid inconsistencies, one must evaluate $G(z, z', s)$, given by (3.1), using a large but finite number of terms ($n < \infty$).

In the context of the approximation defined above (when "n" only assumes finite values) and the L parameter being finite, the following result is valid,

$$\lim_{\epsilon\to 0}\left\{\left(\frac{L}{n\pi\epsilon}\right) sin\left(\frac{n\pi\epsilon}{L}\right)\right\} = 1. \quad (3.15)$$

So, expression (3.14) is,

$$f_n(s) = \left(\frac{4\bar{Q}(s)}{\pi^2 KL}\right) sin\left(\frac{n\pi z_0}{L}\right). \quad (3.16)$$

Substituting (3.16) into (3.11) we have,

$$\varphi_n(s) = \frac{1}{\pi^2 K}\left(\frac{4n\pi\bar{Q}(s)}{n^2\pi^2 + \mu^2 L^2}\right) sin\left(\frac{n\pi z_0}{L}\right), \quad (3.17)$$

in this way, by substituting (3.17) into (3.1), we find a well-defined expression for $G(z, z', s)$,

$$G(z, z', s) = \frac{1}{\pi^2 K}\sum_{n=1}^{\infty'}\left(\frac{4n\pi\bar{Q}(s)}{n^2\pi^2 + \mu^2 L^2}\right) sin\left(\frac{n\pi z_0}{L}\right) cos\left(\frac{n\pi}{L}(z - z')\right), \quad (3.18)$$

where the notation ∞' was used to refer to an arbitrarily large number, however: $\infty' < \infty$.

The expression (3.18), as already commented, is not possible to be calculated exactly (considering every value of "n") without introducing inconsistencies; therefore, that expression must evaluate to a finite number of terms (approximate solution). Expression (3.18) is valid in the range $< 0, L >$ of variable z.

Note that the function Q, which for the time being remains undefined, depends on the source, that is, how the source generates heat over time, and that the function $\bar{Q}$ is the Laplace transform of Q.

3.1. Formal solution of the heat equation. Replacing (3.18) into (2.11) we have,

$$\bar{\Theta}(z,s) = \frac{1}{\pi^2 k}\sum_{n=1}^{\infty'}\left(\frac{4n\pi\bar{Q}(s)}{n^2\pi^2+\mu^2L^2}\right) sin\Big(\frac{n\pi z_0}{L}\Big)\int_0^z dz' cos\Big(\frac{n\pi}{L}(z-z')\Big), \quad (3.19)$$

the same one that, after doing the integration, is like,

$$\bar{\Theta}(z,s) = \frac{1}{\pi^2 k}\sum_{n=1}^{\infty'}\left(\frac{4L\bar{Q}(s)}{n^2\pi^2+\mu^2L^2}\right) sin\left(\frac{n\pi z_0}{L}\right) sin\left(\frac{n\pi}{L}z\right). \quad (3.20)$$

To obtain the function θ from (3.20) the corresponding inverse transformations are applied; so, we wrote,

$$\theta = F_y^{-1}\Big\{F_x^{-1}L^{-1}\bar{\Theta}\Big\}. \quad (3.21)$$

Using the fact that the two operations (Laplace transform and Fourier transform) commute we can also write,

$$\theta = L^{-1}\Big\{F_y^{-1}F_x^{-1}\bar{\Theta}\Big\}. \quad (3.22)$$

The corresponding expression for the value of the above function is written,

$$\theta(t,x,y,z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\Theta}(s,\alpha,\beta,z)e^{-i(\alpha x+\beta y)}d\alpha d\beta ds \quad (3.23)$$

Note, from expression (3.20), that the variables s and z of the function $\bar{\Theta}$ are separated, whereas the variables α and β are coupled through the relation $\mu^2 = \alpha^2+\beta^2+(ms/k)$.

The inverse (double) Fourier transform of $\bar{\Theta}$ is written as,

$$\bar{\theta} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\Theta}(s,\alpha,\beta,z)e^{-i(\alpha x+\beta y)})d\alpha d\beta, \quad (3.24)$$

the one that can be rewritten in terms of the Hankel transform [9],

$$\bar{\theta}(s,r,z) = 2\pi\int_0^{\infty}\rho\bar{\Theta}(s,\rho,z)J_0(\rho r)d\rho, \quad (3.25)$$

where $\rho^2 = \alpha^2+\beta^2$, $r^2 = x^2+y^2$, and J_0 is the Bessel function of the first type and of zero order. So, substituting (3.20) for (3.25) and the result, in turn, substituting it for (3.23), we arrive at the expression,

$$\theta(t,r,z) = \frac{8\pi L}{\pi^2 k}\sum_{n=1}^{\infty'}\left[sin\left(\frac{n\pi z_0}{L}\right) sin\left(\frac{n\pi}{L}z\right)\times\right.$$

$$\left.\times\int_0^{\infty}\rho\ J_0(\rho r)\left\{\Big(\frac{1}{2\pi i}\Big)\int_{\gamma-i\infty}^{\gamma+i\infty}\left(\frac{e^{st}\bar{Q}(s)}{n^2\pi^2+(\rho^2+\frac{ms}{k})L^2}\right)ds\right\}d\rho\ \right] \quad (3.26)$$

where the integral between keys takes place in the complex plane s along a Bromwich line [8]. The expression in (3.26) represented what might be called a formal solution to the temperature increment function, since the function $\bar{Q}(s)$ remains undefined. In the next subsection we consider a specific heat source.

3.1.1. *Solution for a point source of heat.* The complex integral in (3.26) will be calculated for the particularly simple case of a point source that generates a rectangular heat pulse during a time "a"; therefore, we can write for its Laplace transform,

$$\bar{Q}(s) = \frac{1}{s}\Big(1 - e^{-as}\Big). \tag{3.27}$$

So, we can calculate the integral between braces in (3.26),

$$A \equiv \Big(\frac{1}{2\pi i}\Big)\int_{\gamma-i\infty}^{\gamma+i\infty}\left(\frac{e^{st}\bar{Q}(s)}{n^2\pi^2+\rho^2L^2+\frac{mL^2s}{k}}\right)ds =$$

$$= \Big(\frac{1}{2\pi i}\Big)\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{(1-e^{-as})e^{st}}{s(\beta_n+\alpha s)}ds, \tag{3.28}$$

where,

$$\beta_n = n^2\pi^2+\rho^2L^2 \qquad \& \qquad \alpha = \frac{mL^2}{k}.$$

On the other hand, the factor $1\Big/\Big(s(\beta_n+\alpha s)\Big)$ in the integrand in (3.28) we rewrite it as follows,

$$\frac{1}{s(\beta_n+\alpha s)} = \frac{1}{\alpha\Gamma_n}\left(\frac{1}{s}-\frac{1}{s+\Gamma_n}\right), \tag{3.29}$$

where,

$$\Gamma_n = \frac{\beta_n}{\alpha} = \frac{(n^2\pi^2+\rho^2L^2)k}{mL^2}.$$

In this way, we come to the expression,

$$A = \frac{1}{\beta_n}\left(\frac{1}{2\pi i}\right)\int_{\gamma-i\infty}^{\gamma+i\infty}\left(\frac{e^{st}}{s}-\frac{e^{st}}{s+\Gamma_n}-\frac{e^{s(t-a)}}{s}+\frac{e^{s(t-a)}}{s+\Gamma_n}\right)ds. \tag{3.30}$$

The (four) integrals in (3.30), for the cases $t < 0$ and $t > 0$, are well known in the literature dealing with the integration in the complex plane "s" along a Bromwich line [8]. Based on this reference, we have the following result,

$$A = \left(\frac{1}{n^2\pi^2+\rho^2L^2}\right)\times$$

$$\times\left\{S(t) - S(t)\,e^{\frac{(n^2\pi^2+\rho^2L^2)kt}{mL^2}} - S(t-a) + S(t-a)\,e^{\frac{(n^2\pi^2+\rho^2L^2)k(t-a)}{mL^2}}\right\}, \tag{3.31}$$

where S is the Heaviside function. In this way, we arrive at an expression for the temperature increment function,

$$\theta(t,r,z) = \frac{8\pi L}{\pi^2k}\sum_{n=1}^{\infty'}\left[sin\left(\frac{n\pi z_0}{L}\right)sin\left(\frac{n\pi}{L}z\right)\int_0^\infty\left(\frac{\rho J_0(\rho r)}{n^2\pi^2+\rho^2L^2}\right)\times\right.$$

$$\left.\times\left\{S(t)-S(t)\,e^{\frac{(n^2\pi^2+\rho^2L^2)kt}{mL^2}}-S(t-a)+S(t-a)\,e^{\frac{(n^2\pi^2+\rho^2L^2)k(t-a)}{mL^2}}\right\}d\rho\right]. \tag{3.32}$$

The integrals in (3.32), for $n = 1, 2, ..$, can be computed computationally using simple numerical resources, but this is not done here. Note that expression (3.32) checks for $\theta(t = 0^-, r, z) = 0$, as expected.

4. Conclusion

We have considered the heat equation with a point source that generates a heat pulse in a homogeneous medium. A solution was found through an approach that used a specific integral transformation, which enabled the emergence of an integral-differential equation of the Volterra type, linear and non-homogeneous. The integral-differential equation was solved giving rise to an approximate solution to the heat equation. The integrals that appear in expression (3.32) for the solution of the heat equation could easily be calculated using elementary numerical methods, which was not done here. The approach presented could also be applied to other equations in mathematical physics (provided that the exact corresponding equation is not required); for example, the Helmholtz equation [10], [11], [12], [13], [14], [15].

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