A case of spurious quantum entanglement originated by a mathematical property with a non-physical parameter

J. D. Bulnes[1], **F. A. Bonk**[2]
[1]*Grupo de Mecânica Quântica, Informação Quântica e Física Aplicada, Universidade Federal do Amapá, Rod. Juscelino Kubitschek, Km. 2, Jardim Marco Zero, CEP. 68903-419, Macapá, AP, Brazil.*
[2]*Instituto de Ciências Exatas e Tecnologia, Universidade Paulista, Rua Miguel Guidotti, 405, Egisto Ragazzo, CEP.13485-342, Limeira, SP, Brazil.*

E-mail: bulnes@unifap.br

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Abstract

In this paper it's build a 4×4 entangled matrix with the shape of a pseudo-pure matrix, $\rho_\varepsilon = (1-\varepsilon)I_4/4+\varepsilon\rho_1$, in which the matrix ρ_1 also is entangled. The entanglement was identified through the Peres-Horodecki criterion, but such entanglement, as it's shown, it's not physical, but only mathematical.

Keywords: Quantum entanglement, Extended pseudo-pure matrix.

Resumen

En este artículo se construye una matriz entrelazada 4×4 con la forma de una matriz seudopura $\rho_\varepsilon = (1-\varepsilon)I_4/4+\varepsilon\rho_1$ en la cual la matriz ρ_1 también está entrelazada. El entrelazamiento fue identificado a través del criterio de Peres-Horodecki, pero tal entrelazamiento, como se muestra, no es físico, sino únicamente matemático.

Palabras clave: Entrelazamiento cuántico, Matriz pseudo-pura extendida.

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I. INTRODUCTION

On this section, we briefly reviewed the concept of extended pseudo-pure matrix [1], to later construct a particular entangled matrix which includes a matrix that is also entangled.

Considering only the mathematical form (but not the origin) of a pseudo-pure matrix, [2, 3, 4], in the case of 4×4 matrices, we write:

$$\rho_\varepsilon = (1-\varepsilon)\frac{I_4}{4}+\varepsilon\rho_1, \tag{1}$$

in which:
ε is a parameter with value in the interval $<0,1>$, I_4 is the identity matrix and ρ_1 is a density matrix.

A generalization of the kind of the above matrix (as it was performed in [1]) consists of maintaining the form pseudo-pure and including a Hermitian matrix ρ_1^E that has (at least) a negative eigenvalue in the place of the density matrix ρ_1; then we write:

$$\rho_\varepsilon^E = (1-\varepsilon)\frac{I_4}{4}+\varepsilon\rho_1^E. \tag{2}$$

It's evident that if the matrix ρ_1^E is chosen arbitrarily, the corresponding matrix ρ_ε^E not necessarily will be "density matrix" type, then, to assure that ρ_ε^E is always a density matrix, it's necessary to impose some condition.

Considering the fact that ρ_ε^E and ρ_1^E commute and denominating λ_ε and λ_1 to eigenvalues of these matrices defined to the set de eigenvectors in common, one comes to the following relation,

$$\lambda_\varepsilon = (1-\varepsilon)/4 \; + \; \varepsilon\lambda_1. \tag{3}$$

Therefore, imposing that $\lambda_\varepsilon \geq 0$, one gets,

$$\lambda_\varepsilon \geq -(1-\varepsilon)/4\varepsilon\,. \tag{4}$$

The previous result would be useful if we could choose matrices ρ_1^E through their eigenvalues: if the previous relation is satisfied, the corresponding matrix ρ_{ε}^{E} is a density matrix. Among the density matrices thus generated some are entangled, but it is only a mathematical entanglement (and not a physical entanglement), as it was shown in [1].

II. CONSTRUCTING A MATRIX WITH ENTANGLEMENT 'IN WHOLE AND IN PART'

Hereafter extends the context discussed in [1] and then to construct a case of mathematical entanglement. Let us consider two entangled pseudo-pure matrices ρ_{ε} and $\rho_{\varepsilon'}$ that are constructed starting from the same extended pseudo-pure matrix ρ_1^E, having each one distinct value of its parameter, ε and ε', that by now will be considered arbitrary. Then, it can be written,

$$\rho_{\varepsilon} = (1-\varepsilon)\frac{I_4}{4} + \varepsilon\rho_1^E. \tag{5}$$

$$\rho_{\varepsilon'} = (1-\varepsilon')\frac{I_4}{4} + \varepsilon'\rho_1^E. \tag{6}$$

The matrix ρ_1^E is chosen, necessarily, through its eigenvalues, λ, all of them satisfy the condition,

$$\lambda \geq -\min\{(1-\varepsilon)/4\varepsilon,\ (1-\varepsilon')/4\varepsilon'\}. \tag{7}$$

Let's suppose that $\varepsilon > \varepsilon'$; then, there is σ in the open interval $<0,1>$, so that $\varepsilon' = \sigma\varepsilon$, and the Equations (5) and (6) can be suitably combined to obtain:

$$\rho_{\varepsilon'} = (1-\sigma)\frac{I_4}{4} + \sigma\rho_{\varepsilon}, \tag{8}$$

that maintains the pseudo-pure form and where the Hermitian matrix ρ_1^E don't appear anymore; we emphasize that in the expression (8) both $\rho_{\varepsilon'}$ and ρ_{ε} are density matrices.

Hereafter is presented a numerical example of matrices of the type described above. In the open interval $<-60,+55>$, which was numerically identified, are selected randomly several sets of fifteen numbers, $C_{i,j}$, to which we add the element $C_{1,1}$, with fixed value equal to one. One of these sets of numbers was ordered in the following matrix,

$$C = \begin{pmatrix} 1 & -14{,}6958 & -21{,}4681 & -23{,}6227 \\ +18{,}4872 & -29{,}9316 & +39{,}7585 & -53{,}4624 \\ +9{,}1401 & +33{,}7877 & -51{,}0052 & -53{,}6334 \\ -11{,}8384 & +52{,}3318 & +39{,}8948 & +23{,}4070 \end{pmatrix}. \tag{9}$$

The numbers $C_{i,j}$, elements of the previous matrix, will be considered as independent coefficients of the expansion of a matrix ρ_1^E at the basis of Pauli density matrices. Then it's written,

$$\rho_1^E = \frac{1}{4}\sum_{i,j=1}^{4} C_{i,j}\sigma_i \otimes \sigma_j, \tag{10}$$

finding the Hermitian matrix,

$$\rho_1^E = \begin{pmatrix} -2{,}7635 & +9{,}4090-4{,}6067i & -8{,}7438+11{,}1233i & +5{,}2684-18{,}3865i \\ +9{,}4090+4{,}6067i & -2{,}6557 & -20{,}2342+1{,}4927i & +17{,}9874-15{,}6934i \\ -8{,}7438-11{,}1233i & -20{,}2342-1{,}4927i & -8{,}5478 & -16{,}7569+15{,}3407i \\ +5{,}2684+18{,}3865i & +17{,}9874+15{,}6934i & -16{,}7569-15{,}3407i & +14{,}9670 \end{pmatrix}, \tag{11}$$

that has trace equal to one and presents the following eigenvalues: -28,1068, -17,9439, -11,4005, +58,4513; therefore, the matrix ρ_1^E isn't a density matrix. With this matrix, and according to the expressions (5) and (6), and the values $\varepsilon = 0{,}008$ and $\varepsilon' = 0{,}005$, respectively, are calculated the matrices ρ_{ε} and $\rho_{\varepsilon'}$.

$$\rho_{\varepsilon} = \begin{pmatrix} +0{,}2259 & +0{,}0753-0{,}0369i & -0{,}0700+0{,}0890i & +0{,}0421-0{,}1471i \\ +0{,}0753+0{,}0369i & +0{,}2268 & -0{,}1619+0{,}0119i & +0{,}1439-0{,}1255i \\ -0{,}0700-0{,}0890i & -0{,}1619-0{,}0119i & +0{,}1796 & -0{,}1341+0{,}1227i \\ +0{,}0421+0{,}1471i & +0{,}1439+0{,}1255i & -0{,}1341-0{,}1227i & +0{,}3677 \end{pmatrix}, \tag{12}$$

which is a density matrix because it is Hermitian, it has trace equal to one and eigenvalues non-negative: +0,0231, +0,1044, +0,1568, +0,7156.

$$\rho_{\varepsilon'} = \begin{pmatrix} +0{,}2349 & +0{,}0470-0{,}0230i & -0{,}0437+0{,}0556i & +0{,}0263-0{,}0919i \\ +0{,}0470+0{,}0230i & +0{,}2355 & -0{,}1012+0{,}0075i & +0{,}0899-0{,}0785i \\ -0{,}0437-0{,}0556i & -0{,}1012-0{,}0075i & +0{,}2060 & -0{,}0838+0{,}0767i \\ +0{,}0263+0{,}0919i & +0{,}0899+0{,}0785i & -0{,}0838-0{,}0767i & +0{,}3236 \end{pmatrix}, \quad (13)$$

which have all their eigenvalues non-negative: +0,1082, +0,1590, +0,1917, +0,5410.

On the other hand, using the Peres-Horodecki criterion, [5, 6], establishing, in the case of 4×4 matrices, a necessary and sufficient condition for separability, we find that this isn't satisfied, neither by matrix ρ_ε nor the matrix $\rho_{\varepsilon'}$, because their partial transposed matrices, $T_P(\rho_\varepsilon)$ and $T_P(\rho_{\varepsilon'})$, present[1] a negative eigenvalue: -0,1966 and -0,0291, respectively; thus, both matrices are entangled based on the Peres-Horodecki criterion. Moreover, observing the mathematical expression in Equation 8, with $\rho_{\varepsilon'}$ and ρ_ε entangled, we see a matrix that seems to represent correctly certain physical state. According to the general considerations [7], to attribute physical meaning to mathematical objects (in this case matrices) of a physical model it should be possible to put them in correspondence with the physical system considered, but, for that, it should be shown two things: (i) that the matrix ρ_ε in Equation 8 it may be constructed from the application of unitary operations to the thermal equilibrium density matrix, and (ii) that the parameter σ corresponds to the experimental conditions in some specific implementation of NMR quantum computing. In this case, it's not possible to establish such correspondence because the parameter σ was obtained without taking into consideration experimental values, but through the use of a mathematical property of states considered: the state (8) is a result of a mathematical 'trick'.

It still remains to show that the assumption considered, that there are two entangled matrices associated with a matrix ρ_1^E, has justification. For that, we need to show that there are Hermitian matrices ρ_1^E whose eigenvalues can satisfy the relation (7) for two numbers ε and ε' and then seek, on the set of extended pseudo-pure density matrices which may be generated, those that are *entangled*; this was been numerically.

Let's consider[2] the ε – Maps corresponding to values $\varepsilon = 0{,}005$ and $\varepsilon = 0{,}008$, Figures 1 and 2, those that present (when superposed) non-null intersection, for example, to the interval $< -60,+55 >$. Having the Figures non-null intersection it means that the condition in Equation 7 is satisfied: there are numbers which define a single extended matrix ρ_1^E to the two parameter's values, so that the extended pseudo-pure matrices generated are density matrices. The matrix C considered in Equation 9 has, precisely, its elements in an interval in which there is non-null intersection of the two ε – Maps. It is important to point out that, unlike the examples presented in [1], here we have a case of mathematical entanglement due to the fact that the parameter σ doesn't have a physical meaning.

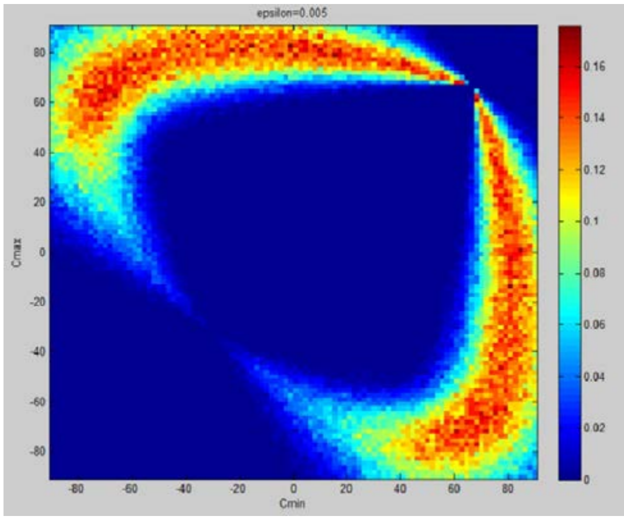

FIGURE 1. ε – Map corresponding to value $\varepsilon = 0{,}005$.

Ten thousand extended matrices had been generated, indirectly, through the definition of their expansion coefficients in a basis of Pauli density matrices, as it's described on Appendix 1. Among the density matrices with the shape of a pseudo-pure matrix that had been generated, the map shows, in a color scale, the distribution and percentage of those that are entangled.

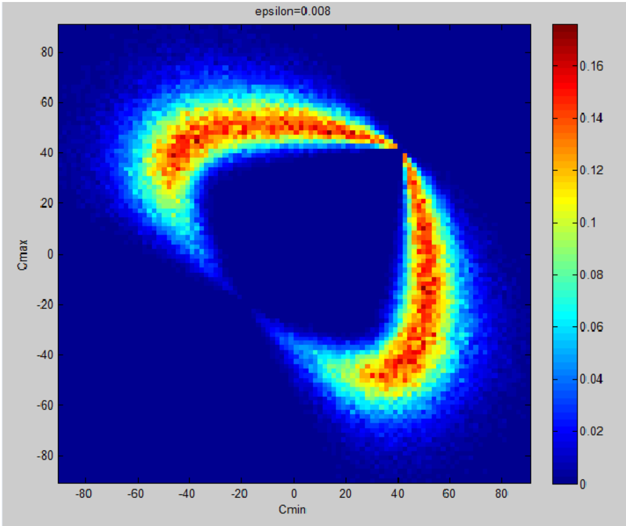

FIGURE 2. ε – Map corresponding to value $\varepsilon = 0{,}008$.

[1] The eigenvalues of $T_P(\rho_\varepsilon)$ are: -0,1966, +0,2993, +0,3554, +0,5419, and the eigenvalues of $T_P(\rho_{\varepsilon'})$ are: -0,0291, +0,2808, +0,3159, +0,4324.

[2] See the appendix 1 for the corresponding definition.

As in Figure 1, here are considered the same scales and limit values in the two coordinate directions.

VI. CONCLUSIONS

It had been constructed a pseudo-pure matrix (Equation 8 or 13) from two extended pseudo-pure matrices (Equations 5, 6) that are entangled according to Peres-Horodecki criterion, showing that the entanglement associated with the pseudo-pure matrix isn't physical, but only mathematical, because it includes a parameter whose value doesn't arise of experimental values but that results of a mathematical 'trick', opposed to the corresponding to a physical state with the same shape.

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APPENDIX 1. DEFINITION OF A $\varepsilon -$ MAP

We presented a definition related with counting and distribution of entangled pseudo-pure matrices in a certain numerical context. Let's call '$\varepsilon -$Map' to a map that determines, in a scale of colors, for a certain value of ε and when the condition $\lambda \geq -(1-\varepsilon)/4\varepsilon$ is satisfied by each matrix ρ_1^E generated, the fraction of pseudo-pure matrices ρ_ε identified by the Peres-Horodecki criterion (in the case of 4×4 matrices and when it doesn't identify separability) which are found in each interval $< C_{\min}, C_{\max} >$ (associated with some point on the map) where are randomly defined, fifteen real numbers $C_{i,j}$ that are considered as independent coefficients of the expansion (in a basis of Pauli density matrices) of a particular extended matrix ρ_1^E, which generates, through (4), a matrix ρ_ε.