Scientia Magna
Vol. 17 (2022), No. 1, 77-84

Leverrier-Takeno and Faddeev-Sominsky algorithms

J. D. Bulnes[1], N. Islam[2] and J. López-Bonilla[3]

[1]Dep. de Ciências Exatas e Tecnologias, Universidade Federal do Amapá
Rod. J. Kubitschek, 68903-419, Macapá, Brazil
E-mail: bulnes@unifap.br
[2]Ramgarh Engineering College, Ramgarh,
Jharkhand 825 101, India
E-mail: nazmul.islam786@gmail.com
[3]ESIME-Zacatenco, Instituto Politécnico Nacional
Edif. 4, 1er. Piso, Col. Lindavista, CP 07738, CDMX, México
E-mail: jlopezb@ipn.mx

Abstract We exhibit representations of the coefficients of the characteristic polynomial of any matrix $A_{n\times n}$, especially in terms of the (exponential) complete Bell polynomials. Besides, we use the Faddeev-Sominsky method to obtain the Lanczos formula for the resolvent of a matrix. We indicate that the Newton's recurrence formula can be solved via the inversion of a triangular matrix.

Keywords Faddeev-Sominsky's algorithm, Cayley-Hamilton- Frobenius theorem, Leverrier-Takeno's procedure, (Exponential) complete Bell polynomials.
2010 Mathematics Subject Classification 15Axx, 97H60, 15A09, 11C20, 40C05.

§1. Introduction

For an arbitrary matrix $A_{n\times n} = (A^i{}_j)$ its characteristic polynomial [1–3]:

$$p(\lambda) \equiv \lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n, \tag{1}$$

can be obtained, through several procedures [1,4–8], directly from the condition $det\ (A^i{}_j - \lambda\delta^i_j) = 0$. The method of Leverrier-Takeno [4, 9–13] is a simple and interesting technique to construct (1) based in the traces of the powers A^r, $r = 1, ..., n$.

On the other hand, it is well known that an arbitrary matrix satisfies its characteristic equation, which is the Cayley-Hamilton-Frobenius identity [1–3, 14]:

$$A^n + a_1A^{n-1} + ...a_{n-1}A + a_nI = 0. \tag{2}$$

If A is non-singular (that is, $det\ A \neq 0$), then from (2) we obtain its inverse matrix:

$$A^{-1} = -\frac{1}{a_n}\left(A^{n-1} + a_1A^{n-2} + ... + a_{n-1}I\right), \tag{3}$$

where $a_n \neq 0$ because $a_n = (-1)^n \; det \; A$.

Faddeev-Sominsky [15–24] proposed an algorithm to determine A^{-1} in terms of A^r and their traces, which is equivalent [23] to the Cayley-Hamilton-Frobenius theorem (2) plus the Leverrier-Takeno's method to construct the characteristic polynomial of a matrix A. In Sec. 2, we use the Faddeev-Sominsky's procedure to deduce the Lanczos expression [25] for the resolvent of A [20, 21, 26, 27], that is, the Laplace transform of $exp \; (tA)$ [28].

§2. Leverrier-Takeno technique

If we define the quantities:

$$a_0 = 1, \quad s_k = tr \; A^k \quad k = 1, 2, ..., n \tag{4}$$

then the approach of Leverrier-Takeno [4, 9–13] implies (1) wherein the a_i are determined with the Newton's recurrence relation:

$$r a_r + s_1 a_{r-1} + s_2 a_{r-2} + ... + s_{r-1} a_1 + s_r = 0, \quad r = 1, 2, ..., n \tag{5}$$

therefore:

$$\begin{gathered} a_1 = -s_1, \quad 2! \; a_2 = (s_1)^2 - s_2, \quad 3! \; a_3 = -(s_1)^3 + 3 s_1 s_2 - 2 s_3, \\ 4! \; a_4 = (s_1)^4 - 6 (s_1)^2 s_2 + 8 s_1 s_3 + 3 (s_2)^2 - 6 s_4, \\ 5! \; a_5 = -24 s_5 - (s_1)^5 + 10 (s_1)^3 s_2 - 20 (s_1)^2 s_3 - 15 (s_2)^2 s_1 + 30 s_1 s_4 + 20 s_2 s_3, \; etc, \end{gathered} \tag{6}$$

in particular, $det \; A = (-1)^n a_n$, that is, the determinant of any matrix only depends on the traces s_r, which means that A and its transpose have the same determinant. In [29–31] we find the general expression:

$$a_k = \frac{(-1)^k}{k!} \begin{vmatrix} s_1 & k-1 & 0 & ... & 0 \\ s_2 & s_1 & k-2 & ... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ s_{k-1} & s_{k-2} & ... & ... & 1 \\ s_k & s_{k-1} & ... & ... & s_1 \end{vmatrix}, \qquad k = 1, ..., n, \tag{7}$$

which allows reproduce the values (6).

We can exhibit a relation to determine the coefficients a_j via the complete Bell polynomials [8, 32–40], in fact, we have the following representation [8]:

$$m! \; a_m = Y_m \Big(-0! s_1, -1! s_2, -2! s_3, -3! s_4, ..., -(m-2)! s_{m-1}, -(m-1)! s_m \Big). \tag{8}$$

such that [37, 41]:

$$
Y_m(x_1, x_2, ..., x_m) = \begin{vmatrix} \binom{m-1}{0}x_1 & \binom{m-1}{1}x_2 & ... & \binom{m-1}{m-2}x_{m-1} & \binom{m-1}{m-1}x_m \\ -1 & \binom{m-2}{0}x_1 & ... & \binom{m-2}{m-3}x_{m-2} & \binom{m-2}{m-2}x_{m-1} \\ 0 & -1 & ... & \binom{m-3}{m-4}x_{m-3} & \binom{m-3}{m-3}x_{m-2} \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \binom{1}{0}x_1 & \binom{1}{1}x_2 \\ 0 & 0 & ... & -1 & \binom{0}{0}x_1 \end{vmatrix}, \tag{9}
$$

therefore:

$$
\begin{gathered}
Y_0 = 1, \;\; Y_1 = x_1, \;\; Y_2 = x_1^2 + x_2, \\
Y_3 = x_1^3 + 3\,x_1x_3 + x_3, \;\; Y_4 = x_1^4 + 6\,x_1^2x_2 + 4\,x_1x_3 + 3\,x_2^2 + x_4, \\
Y_5 = x_1^5 + 10\,x_1^3x_2 + 10\,x_1^2x_3 + 15\,x_1x_2^2 + 5\,x_1x_4 + 10\,x_2x_3 + x_5, \;\; ...
\end{gathered} \tag{10}
$$

We see that (8) and (10) imply (6) if we employ $x_1 = -s_1$, $x_2 = -s_2$, $x_3 = -2s_3$, $x_4 = -6s_4$, $x_5 = -24s_5$, ... It is simple to prove that (9) with $x_k = -(k-1)!\, s_k$ gives (7), thus the coefficients of the characteristic polynomial (1) are generated by the (exponential) complete Bell polynomials.

In the Newtons formula (5) the quantities s_r are known, and the a_j are solutions of the triangular linear system [42–44]:

$$
A_{n\times n}\,(a_j)_{n\times 1} \equiv
\begin{pmatrix} 1 & 0 & 0 & ... & ... & 0 \\ s_1 & 2 & 0 & ... & ... & 0 \\ s_2 & s_1 & 3 & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & 0 \\ s_{n-1} & s_{n-2} & s_{n-3} & ... & s_1 & n \end{pmatrix}
\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ ... \\ ... \\ ... \\ a_n \end{pmatrix} = -\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ ... \\ ... \\ ... \\ s_n \end{pmatrix}, \tag{11}
$$

then:

$$
\begin{pmatrix} a_1 \\ ... \\ ... \\ a_n \end{pmatrix} = -A^{-1} \begin{pmatrix} s_1 \\ ... \\ ... \\ s_n \end{pmatrix}, \tag{12}
$$

which gives the opportunity to invert a triangular matrix via interesting algorithms applying the Faddeev-Sominsky method [15–24], matrix multiplication [45, 46] or binomial series [47].

§3. Faddeev-Sominsky procedure

The Faddeev-Sominsky's technique to obtain A^{-1} is a sequence of algebraic computations on the powers A^r and their traces, in fact, this algorithm is given via the following instructions:

$$
\begin{array}{ccc}
B_1 = A, & q_1 = \mathrm{tr}\ B_1, & C_1 = B_1 - q_1 I, \\
B_2 = C_1\ A, & q_2 = \dfrac{1}{2}\ \mathrm{tr}\ B_2, & C_2 = B_2 - q_2 I, \\
\dots & \dots & \dots \\
\dots & \dots & \dots \\
B_{n-1} = C_{n-2}\ A, & q_{n-1} = \dfrac{1}{n-1}\ \mathrm{tr}\ B_{n-1}, & C_{n-1} = B_{n-1} - q_{n-1} I, \\
B_n = C_{n-1}\ A, & q_n = \dfrac{1}{n}\ \mathrm{tr}\ B_n, &
\end{array} \tag{13}
$$

then:

$$A^{-1} = \frac{1}{q_n}\ C_{n-1}. \tag{14}$$

For example, if we apply (13) for $n = 4$, then it is easy to see that the corresponding q_r imply (6) with $q_j = -a_j$, and besides (14) reproduces (3). By mathematical induction one can prove that (13) and (14) are equivalent to (3), (4) and (5), showing [23] thus that the Faddeev-Sominsky's technique has its origin in the Leverrier-Takeno method plus the Cayley-Hamilton-Frobenius theorem.

From (13) we can see that [26]:

$$C_k = A^k + a_1 A^{k-1} + a_2 A^{k-2} + \dots + a_{k-1} A + a_k I, \quad k = 1, 2, \dots, n-1, \quad C_n = B_n - q_n I = 0, \tag{15}$$

and for $k = n - 1$:

$$C_{n-1} = A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \dots + a_{n-2} A + a_{n-1} I \overset{(3)}{=} -a_n A^{-1}$$

in harmony with (14) because $a_n = -q_n$. The property $C_n = 0$ is equivalent to (2); if A is singular, the process (13) gives the adjoint matrix of A [2,3,16], in fact, $Adj\ A = (-1)^{n+1} C_{n-1}$.

If the roots of (1) have distinct values, then the Faddeev-Sominsky's algorithm allows obtain the corresponding eigenvectors of A [6]:

$$A\vec{u}_k = \lambda_k \vec{u}_k, \qquad k = 1, 2, \dots, n, \tag{16}$$

because for a given value of k, each column of:

$$Q_k \equiv \lambda_k^{n-1} I + \lambda_k^{n-2} C_1 + \dots + C_{n-1}. \tag{17}$$

satisfies (16) [16, 18, 27], and therefore all columns of Q_k are proportional to each other, that is, $rank\ Q_k = 1$ [18]; we note that $Q_k = Q(\lambda_k)$ with the participation of the matrix:

$$Q(z) \equiv z^{n-1}I + z^{n-2}C_1 + z^{n-3}C_2 + ... + zC_{n-2} + C_{n-1}. \tag{18}$$

By synthetic division of two polynomials [1]:

$$\frac{p(z)}{z-\lambda} = \sum_{r=0}^{n-1} \left(\lambda^r + a_1\lambda^{r-1} + a_2\lambda^{r-2} + ... + a_{r-1}\lambda + a_r \right) z^{n-1-r},$$

then under the change $\lambda \to A$ we obtain the Lanczos expression [25] for the resolvent of a matrix [20, 21, 26, 27]:

$$\frac{1}{zI - A} = \frac{1}{p(z)} \sum_{r=0}^{n-1} z^{n-1-r}\ C_r = \frac{Q(a)}{p(z)}, \tag{19}$$

if A is non-singular, then [19] for $z = 0$ implies (14). McCarthy [48] used (19) and the Cauchy's integral theorem in complex variable to show the Cayley-Hamilton-Frobenius identity indicated in (2); the relation (19) is the Laplace transform of $exp\ (tA)$ [28].

On the other hand, Sylvester [49–52] obtained the following interpolating definition of $f(A)$:

$$f(A) = \sum_{j=1}^{n} f(\lambda_j) \prod_{k \neq j} \frac{A - \lambda_k I}{\lambda_j - \lambda_k}, \tag{20}$$

which is valid if all eigenvalues are different from each other. Buchheim [53] generalized (20) to multiple proper values using Hermite interpolation, thereby giving the first completely general definition of a matrix function. From (19) and (20) for $f(s) = 1/(z-s)$ we deduce the properties:

$$Q(z) = \sum_{j=1}^{n} \prod_{k=1, k\neq j}^{n} \frac{z - \lambda_k}{\lambda_j - \lambda_k} \left(A - \lambda_k I \right), \qquad Q_j = \prod_{k=1, k\neq j}^{n} \left(A - \lambda_k I \right)$$

$$Q_j\ \vec{u}_j = \prod_{k=1, k\neq j}^{n} \left(\lambda_j - \lambda_k \right) \vec{u}_j, \tag{21}$$

hence the eigenvectors of A showed in (16) also are proper vectors of the matrices Q_j. Besides, from (16) and (21):

$$A\ Q_j\ \vec{u}_j = \prod_{k=1, k\neq j}^{n} \left(\lambda_j - \lambda_k \right) \lambda_j \vec{u}_j = \lambda_j\ Q_j\ \vec{u}_j, \qquad A\ Q_j = \lambda_j\ Q_j \tag{22}$$

that is, each column of Q_j is eigenvector of A with proper value λ_j. The resolvent (19) implies the relation $(A - zI)Q(z) = -p(z)I$, then $(A - \lambda_k I)\ Q(\lambda_k) = -p(\lambda_k)I = 0$ in according with (22).

From the Sylvester's formula (20) with $f(z) = p(z)$ we obtain $p(A) = 0$, which is the Cayley-Hamilton-Frobenius theorem indicated in (2). If $f(z) = e^{tz}$, then (20) allows to construct $exp\ (tA)$ that, in particular, is valuable to determine the motion of classical charged particles into a homogeneous electromagnetic field, and to integrate the Frenet-Serret equations with constant curvatures [54]. In [51, 55] we find that the book of Frazer-Duncan-Collar [56] emphasizes the important role of the matrix exponential in solving differential equations and was the first to employ matrices as an engineering tool, and indeed the first book to treat matrices as a branch of applied mathematics.

References

[1] C. Lanczos. Applied analysis. Dover, New York, 1988.

[2] L. Hogben. Handbook of linear algebra. Chapman & Hall / CRC Press, London, 2006.

[3] R. A. Horn, Ch. R. Johnson. Matrix analysis. Cambridge University Press, 2013.

[4] H. Wayland. Expansion of determinantal equations into polynomial form. Quart. Appl. Math. 2 (1945), 277-306.

[5] A. S. Householder, F. L. Bauer. On certain methods for expanding the characteristic polynomial. Numerische Math. 1 (1959), 29-37.

[6] J. H. Wilkinson. The algebraic eigenvalue problem. Clarendon Press, Oxford, 1965.

[7] D. Lovelock, H. Rund. Tensors, differential forms, and variational principles. John Wiley and Sons, New York, 1975.

[8] J. López-Bonilla, S. Vidal-Beltrán, A. Zúñiga-Segundo. Characteristic equation of a matrix via Bell polynomials. Asia Mathematika 2 (2018), no. 2, 49-51.

[9] U. J. J. Leverrier. Sur les variations séculaires des éléments elliptiques des sept planétes principales. J. de Math. Pures Appl. Série 1, 5 (1840), 220-254.

[10] A. N. Krylov. On the numerical solution of the equation, that in technical problems, determines the small oscillation frequencies of material systems. Bull. de l'Acad. Sci. URSS 7 (1931), no. 4, 491-539.

[11] H. Takeno. A theorem concerning the characteristic equation of the matrix of a tensor of the second order. Tensor NS 3 (1954), 119-122.

[12] E. B. Wilson, J. C. Decius, P. C. Cross. Molecular vibrations. Dover, New York, 1980, 216-217.

[13] I. Guerrero-Moreno, J. López-Bonilla, J. Rivera-Rebolledo. Leverrier-Takeno coefficients for the characteristic polynomial of a matrix. J. Inst. Eng. (Nepal) 8 (2011), no. 1-2, 255-258.

[14] Ch. A. McCarthy. The Cayley-Hamilton theorem. Amer. Math. Monthly 8 (1975), no. 4, 390-391.

[15] . D. K. Faddeev, I. S. Sominsky. Collection of problems on higher algebra, Moscow, 1949.

[16] . V. N. Faddeeva. Computational methods of linear algebra. Dover, New York, 1959.

[17] . D. K. Faddeev. Methods in linear algebra. W. H. Freeman, San Francisco, USA, 1963.

[18] . J. C. Gower. A modified Leverrier-Faddeev algorithm for matrices with multiple eigenvalues. Linear Algebra and its Applications 31 (1980), no. 1, 61-70.

[19] G. Helmberg, P. Wagner, G. Veltkamp. On Faddeev- Leverrier's method for the computation of the characteristic polynomial of a matrix and of eigenvectors. Linear Algebra and its Applications 185 (1993), 219-233.

[20] Shui-Hung Hou. On the Leverrier-Faddeev algorithm. Electronic Proc. of Asia Tech. Conf. in Maths. (1998).

[21] Shui-Hung Hou. A simple proof of the Leverrier-Faddeev characteristic polynomial algorithm. SIAM Rev. 40 (1998), no. 3, 706-709.

[22] J. López-Bonilla, J. Morales, G. Ovando, E. Ramírez. Leverrier-Faddeev's algorithm applied to spacetimes of class one. Proc. Pakistan Acad. Sci. 43 (2006), no. 1, 47-50.

[23] J. H. Caltenco, J. López-Bonilla, R. Peña-Rivero. Characteristic polynomial of A and Faddeev's method for A-1. Educatia Matematica 3 (2007), no. 1-2, 107-112.

[24] J. López-Bonilla, H. Torres-Silva, S. Vidal-Beltrán. On the Faddeev-Sominsky's algorithm. World Scientific News 106, 2018, 238-244.

[25] C. Lanczos. An iteration method for the solution of the eigenvalue problem of linear differential and integral operators. J. of Res. Nat. Bureau Stand. 45 (1950), no. 4, 255-282.

[26] B. Hanzon, R. Peeters. Computer algebra in systems theory. Dutch Institute of Systems and Control, Course Program, 1999-2000.

[27] R. Cruz-Santiago, J. López-Bonilla, S. Vidal-Beltrán. On eigenvectors associated to a multiple eigenvalue. World Scientific News 100 (2018), 248-253.

[28] J. López-Bonilla, D. Romero-Jiménez, A. Zaldívar- Sandoval. Laplace transform of matrix exponential function. Prespacetime J. 6 (2015), no. 12, 1410-1413.

[29] L. S. Brown. Quantum field theory. Cambridge University Press, 1994.

[30] T. L. Curtright, D. B. Fairlie. A Galileon primer. arXiv: 1212.6972v1 [hep-th], 31 Dec. 2012.

[31] J. López-Bonilla, R. López-Vázquez, S. Vidal-Beltrán. An alternative to Gower's inverse matrix. Scientific News 102 (2018), 166-172.

[32] J. Riordan. Combinatorial identities. John Wiley and Sons, New York, 1968.

[33] L. Comtet. Advanced combinatorics. D. Reidel Pub., Dordrecht, Holland, 1974.

[34] D. A. Zave. A series expansion involving the harmonic numbers. Inform. Process. Lett. 5 (1976), no. 3, 75-77.

[35] X. Chen, W. Chu. The Gauss ${}_2F_1(1)$-summation theorem and harmonic number identities. Integral Transforms and Special Functions 20 (2009), no. 12, 925-935.

[36] J. Quaintance, H. W. Gould. Combinatorial identities for Stirling numbers. World Scientific, Singapore, 2016.

[37] J. López-Bonilla, R. López-Vázquez, S. Vidal-Beltrán. Bell polynomials. Prespacetime J. 9 (2018), no. 5, 451-453.

[38] J. López-Bonilla, S. Vidal-Beltrán, A. Zúñiga-Segundo. On certain results of Chen and Chu about Bell polynomials. Prespacetime J. 9 (2018), no. 7, 584-587.

[39] J. López-Bonilla, S. Vidal-Beltrán, A. Zúñiga-Segundo. Some applications of complete Bell polynomials. World Eng. & Appl. Sci. J. 9 (2018), no. 3, 89-92.

[40] D. F. Connon. Various applications of the (exponential) complete Bell polynomials. http://arxiv.org/ftp/arxiv/papers/1001/1001.2835.pdf 16 Jan 2010.

[41] W. P. Johnson. The curious history of Faà di Bruno's formula. The Math. Assoc. of America 109 (2002), 217-234.

[42] L. Csanky. Fast parallel matrix inversion algorithm. SIAM J. Comput. 5 (1976), 618-623.

[43] G. Wang, Y. Wei, S. Qiao. Generalized inverses: Theory and computations. Springer, Singapore, 2018.

[44] J. López-Bonilla, A. Lucas-Bravo, S. Vidal-Beltrán. Newton's formula and the inverse of a triangular matrix. Studies in Nonlinear Sci. 4 (2019), no. 2, 17-18.

[45] http://mobiusfunction.wordpress.com/2010/08/07/the-inverse-of-a-triangular-matrix/

[46] J. López-Bonilla, I. Miranda-Sánchez. Inverse of a lower triangular matrix. Studies in Nonlinear Sci. 5 (2020), no. 4, 57-58.

[47] http://mobiusfunction.wordpress.com/2010/12/08/the-inverse-of-triangular-matrix-as-a-binomial-series/

[48] Ch. A. McCarthy. The Cayley-Hamilton theorem. Amer. Math. Monthly 8 (1975), no. 4, 390-391.

[49] J. J. Sylvester. On the equation to the secular inequalities in the planetary theory. Phil. Mag. 16 (1883), 267-269.

[50] A. Buchheim. On the theory of matrices. Proc. London Math. Soc. 16 (1884), 63-82.

[51] N. J. Higham. Functions of matrices: Theory and computation. SIAM, Philadelphia, USA, 2008.

[52] N. J. Higham. Sylvester's influence on applied mathematics. Maths. Today 50 (2014), no. 4, 202-206.

[53] A. Buchheim. An extension of a theorem of Professor Sylvester's relating to matrices. The London, Edinburgh, and Dublin Phil. Mag. and J. of Sci. 22 (1886), no. 135, 173-174.

[54] C. Aguilar, B. E. Carvajal, J. López-Bonilla. A study of matrix exponential function. Siauliai Math. Seminar (Lithuania) 5 (2010), no. 13, 5-17.

[55] A. R. Collar. The first fifty years of aeroelasticity. Aerospace (Royal Aeronautical Soc. Journal) 5 (1978), 12-20.

[56] R. A. Frazer, W. J. Duncan, A. R. Collar. Elementary matrices and some applications to dynamics and differential equations. Cambridge University Press, 1938.