

ON THE PARTITION FUNCTION

J. D. BULNES, T. KIM, J. LÓPEZ-BONILLA, AND S. VIDAL-BELTRÁN

ABSTRACT. In this paper, we exhibit short deductions of the Jha and Malenfant expressions for the partition function, and we show several connections between the Euler, divisor and partition functions via the partial Bell polynomials and Hessenberg determinants. Besides, we show that the Gandhi's recurrence relation for colour partitions implies the Osler-Hassen-Chandrupatla's expression for the divisor function.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 11A25, 11Axx, 11B37, 11B75, 11P83, 97Fxx, 97F60.

KEYWORDS AND PHRASES. Partition function, Malenfant and Jha formulas, Bell polynomials, q -analysis, Colour partitions, Hessenberg determinant, Sum of divisors function.

1. INTRODUCTION

We know the following result [11, 7, 19, 5, 12] for the partition function $p(n)$:

$$(1) \quad \sum_{n=0}^{\infty} p(n)t^n = \frac{1}{\sum_{r=0}^{\infty} a_r t^r} = \frac{1}{E(q)} \equiv \frac{1}{(q; q)_{\infty}}, \quad p(0) = a_0 = 1,$$

where:

$$(2) \quad a_j = \begin{cases} 0, & \text{if } j \neq \frac{N}{2}(3N+1), \\ (-1)^N, & \text{if } j = \frac{N}{2}(3N+1), \end{cases} \quad N = 0, \pm 1, \pm 2, \dots,$$

and $E(q)$ is the Euler function; therefore [4, 2, 21, 15]:

$$(3) \quad p(n) = \frac{1}{n!} \sum_{k=0}^n (-1)^k k! B_{n,k}(1! a_1, 2! a_2, \dots, (n-k+1)! a_{n-k+1!}),$$

expression recently deduced by Jha [15], involving the partial Bell polynomials [4, 15, 6, 16], with the recurrence relation [19, 2, 18]:

$$(4) \quad \sum_{k=0}^n a_k p(n-k) = 0,$$

discovered by MacMahon [7, 19, 18].

On the other hand, from [23, 10] we have that relations type (1) are equivalent to the following Hessenberg determinant:

$$(5) \quad p(n) = (-1)^n \begin{vmatrix} a_1 & a_0 & 0 & 0 & \dots & \dots & 0 \\ a_2 & a_1 & a_0 & 0 & \dots & \dots & 0 \\ a_3 & a_2 & a_1 & a_0 & \dots & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \dots & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots & \dots & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots & \dots & a_0 \\ a_n & a_{n-1} & a_{n-2} & \dots & \dots & \dots & a_1 \end{vmatrix},$$

obtained by Malenfant [18].

Remark.- The definition (2) gives the values:

$$(6) \quad a_j = \begin{cases} 1, & j = 0, 5, 7, 22, 26, 51, 57, 92, 100, 145, 155, \dots \\ 0 & \text{otherwise.} \\ -1, & j = 1, 2, 12, 15, 35, 40, 70, 77, 117, 126, 176, \dots \end{cases}$$

and from (1):

$$(7) \quad E(q) = \sum_{n=0}^{\infty} \frac{E^{(n)}(0)}{n!} q^n, \quad E^{(n)}(0) = n! a_n.$$

2. APPLICATIONS OF (1), (3), (4) AND (5)

As examples of similar expressions to (1), (3), (4) and (5), we know the results [12]:

$$(8) \quad (x; q)_n = \sum_{k=0}^n (-1)^k \binom{n}{k}_q q^{\binom{k}{2}} x^k, \quad \frac{1}{(x; q)_{n+1}} = \sum_{k=0}^{\infty} \binom{n+k}{k}_q x^k,$$

then it is immediate the relation:

$$(9) \quad \sum_{k=0}^{\infty} \frac{1}{(q; q)_k} x^k = \frac{1}{\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q; q)_k} x^k}, \quad (q; q)_0 = 1,$$

therefore [4, 2, 21]:

$$(10) \quad \frac{1}{(q; q)_n} = \frac{1}{n!} \sum_{k=1}^n (-1)^k k! B_{n,k} \times \\ \times \left(-\frac{1! q^{\binom{1}{2}}}{(q; q)_1}, \frac{2! q^{\binom{2}{2}}}{(q; q)_2}, -\frac{3! q^{\binom{3}{2}}}{(q; q)_3}, \frac{4! q^{\binom{4}{2}}}{(q; q)_4}, \dots, \frac{(-1)^{n-k+1} (n-k+1)! q^{\binom{n-k+1}{2}}}{(q; q)_{n-k+1}} \right),$$

involving the partial Bell polynomials, with the recurrence property:

$$(11) \quad \sum_{k=0}^n \frac{(-1)^k q^{\binom{k}{2}}}{(q; q)_k (q; q)_{n-k}} = 0, \quad n \geq 1.$$

We can apply to (10) the Birmajer-Gil-Weiner's inversion process [3] to obtain:

$$(12) \quad \frac{(-1)^k q^{\binom{k}{2}}}{(q; q)_n} = \frac{1}{n!} \sum_{k=1}^n (-1)^k k! B_{n,k} \left(\frac{1!}{(q; q)_1}, \frac{2!}{(q; q)_2}, \frac{3!}{(q; q)_3}, \frac{4!}{(q; q)_4}, \dots, \frac{(n-k+1)!}{(q; q)_{n-k+1}} \right).$$

Besides, from [23, 10] we have that relations type (9) are equivalent to the following Hessenberg determinant:

$$(13) \quad \frac{q^{\binom{n}{2}}}{(q; q)_n} = \begin{vmatrix} \frac{1}{(q; q)_1} & 1 & 0 & 0 & \dots & \dots & 0 \\ \frac{1}{(q; q)_2} & \frac{1}{(q; q)_1} & 1 & 0 & \dots & \dots & 0 \\ \frac{1}{(q; q)_3} & \frac{1}{(q; q)_2} & \frac{1}{(q; q)_1} & 1 & \dots & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \dots & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots & \dots & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots & \dots & 1 \\ \frac{1}{(q; q)_n} & \frac{1}{(q; q)_{n-1}} & \frac{1}{(q; q)_{n-2}} & \dots & \dots & \dots & \frac{1}{(q; q)_1} \end{vmatrix},$$

Similarly, Jha [13] deduced the following connection between the sum of divisors function and the numbers (6):

$$(14) \quad (n-1)! \sigma(n) = \sum_{k=1}^n (-1)^k (k-1)! B_{n,k} \left(1!a_1, 2!a_2, \dots, (n-k+1)! a_{n-k+1} \right)$$

therefore, its inverse is given by:

$$(15) \quad n! a_n = \sum_{k=1}^n (-1)^k B_{n,k} \left(0! \sigma(1), 1! \sigma(2), 2! \sigma(3), \dots, (n-k)! 0! \sigma(n-k+1) \right).$$

The r-coloured partition function $p_r(n)$ is defined in terms of the Euler function [8, 14]:

$$(16) \quad [E(q)]^r = \sum_{n=0}^{\infty} p_r(n) q^n,$$

for example, if $r = 0, 1$ then (7) and (16) imply the values:

$$(17) \quad p_0(n) = \begin{cases} 1, & n = 0 \\ 0, & n \geq 1 \end{cases}, \quad p_1(n) = a_n.$$

From (1) and (16):

$$(18) \quad \sum_{n=0}^{\infty} p_r(n) q^n = \frac{1}{[\sum_{k=0}^{\infty} p(k) q^k]^r},$$

thus [10]:

$$(19) \quad \begin{pmatrix} rp(1) & 1 & 0 & \dots & 0 \\ 2rp(2) & (r+1)p(1) & 2 & \dots & 0 \\ 3rp(3) & (2r+1)p(2) & (r+2)p(1) & \dots & 0 \\ \cdot & (3r+1)p(3) & (2r+2)p(2) & \dots & \cdot \\ \cdot & \cdot & (3r+2)p(3) & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ nrp(n) & ((n-1)r+1)p(n-1) & ((n-2)r+2)p(n-2) & \dots & (r+n-1)p(1) \end{pmatrix},$$

$$= (-1)^n n! p_r(n), \quad n \geq 1,$$

with its inverse [15, 13]:

$$(20) \quad p(n) = \sum_{r=0}^n (-1)^r \binom{n+1}{r+1} p_r(n), \quad n \geq 0.$$

For example, if $n = 1, 2, 3, 4$ then (19) implies the relations:

$$(21) \quad \begin{aligned} p_r(1) &= -r, & p_r(2) &= \frac{r}{2!}(r-3), & p_r(3) &= -\frac{r}{3!}(r-1)(r-8), \\ p_r(4) &= \frac{r}{4!}(r-1)(r-3)(r-14). \end{aligned}$$

Johnson [12] exhibits the following connection between the Euler and divisors functions:

$$\begin{aligned} \sum_{n=1}^{\infty} \sigma(n)q^n &= -q \frac{d}{dq} \log(q; q)_{\infty} = -\frac{q}{E(q)} \frac{d}{dq} E(q) \stackrel{(7)}{=} -\frac{1}{E(q)} (a_1q + 2a_2q^2 + 3a_3q^3 + \dots) \\ &\stackrel{(1)}{=} - \left[\sum_{k=0}^{\infty} p(k) q^k \right] \left[\sum_{m=0}^{\infty} m a_m q^m \right] \stackrel{[19]}{=} - \sum_{n=1}^{\infty} q^n \left[\sum_{j=0}^n p(j) (n-j) a_{n-j} \right], \end{aligned}$$

where we can apply (4) to obtain the formula:

$$(22) \quad \sigma(n) = \sum_{k=1}^n k a_{n-k} p(k) = - \sum_{k=1}^n k a_k p(n-k),$$

deduced by Osler-Hassen-Chandrupatla [19]; with (4) and (22) it is immediate the known expression:

$$(23) \quad p(n) = \frac{1}{n} \sum_{j=0}^{n-1} p(j) \sigma(n-j), \quad n \geq 1,$$

finally, we indicate the interesting recurrence relation [19]:

$$(24) \quad \sigma(n) = -n a_n - \sum_{k=1}^{n-1} a_{n-k} \sigma(k), \quad n \geq 2,$$

and the identity [13]:

$$(25) \quad B_{n,k} \left(1! a_1, 2! a_2, \dots, (n-k+1)! a_{n-k+1} \right) = \frac{n!}{k!} \sum_{r=1}^k (-1)^{k-r} \binom{k}{r} p_r(n).$$

It is simple to show that (14) and (25) imply the relationship [13]:

$$(26) \quad \sigma(n) = n \sum_{k=1}^n \frac{(-1)^k}{k} \binom{n}{k} p_k(n), \quad n \geq 1,$$

Remark.- From (19) for $r = 9, n = 4$:

$$p_9(4) = \frac{1}{4!} \begin{vmatrix} 9 & 1 & 0 & 0 \\ 36 & 10 & 2 & 0 \\ 81 & 38 & 11 & 3 \\ 180 & 84 & 40 & 12 \end{vmatrix} = -90 \equiv 0 \pmod{5},$$

in agreement with the congruence $p_9(5m + 4) \equiv 0 \pmod{5}$ indicated by Forbes [14]; besides, from (19) with $r = -1, n = 5$:

$$p_{-1}(5) = \frac{1}{5!} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 4 & 0 & 2 & 0 & 0 \\ 9 & -2 & 1 & 3 & 0 \\ 20 & -6 & 0 & 2 & 4 \\ 35 & -15 & -3 & 2 & 3 \end{vmatrix} = 7 = p(5),$$

which it is correct because $p_{-1}(n) = p(n)$. We note that $p_{24}(n - 1)$ is the Ramanujan's τ -function.

3. RELATIONSHIPS BETWEEN THE SUM OF DIVISORS AND PARTITION FUNCTIONS VIA DETERMINANTS

Gould [10] exhibits the following result:

$$(27) \quad \text{If} \quad e_n = \sum_{j=0}^n b_j^{(n)} c_{n-j}, \quad n \geq 0, \quad b_0^{(k)} \neq 0,$$

then:

$$(28) \quad c_n = \frac{1}{\prod_{r=0}^n b_0^{(r)}} \begin{vmatrix} e_n & b_1^{(n)} & b_2^{(n)} & b_3^{(3)} & \dots & b_n^{(n)} \\ e_{n-1} & b_0^{(n-1)} & b_1^{(n-1)} & b_2^{(n-1)} & \dots & b_{n-1}^{(n-1)} \\ e_{n-2} & 0 & b_0^{(n-2)} & b_1^{(n-2)} & \dots & b_{n-2}^{(n-2)} \\ e_{n-3} & 0 & 0 & b_0^{(n-3)} & \dots & b_{n-3}^{(n-3)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ e_0 & 0 & 0 & \dots & \dots & b_0^{(0)} \end{vmatrix},$$

with interesting applications.

a). Jha [15, 13] obtained the relation (20) with the structure (27):

$$(29) \quad \frac{p(n)}{(n+1)!} = \sum_{j=0}^n \frac{p_{n-j}(n)}{j!} \frac{(-1)^{n-j}}{(n-j+1)!}, \quad e_n = \frac{p(n)}{(n+1)!},$$

$$b_j^{(n)} = \frac{p_{n-j}(n)}{j!}, \quad c_{n-j} = \frac{(-1)^{n-j}}{(n-j+1)!},$$

then (28) and (29) imply the connection:

$$\frac{(-1)^n}{(n+1)!} = \frac{1}{\prod_{r=0}^n p_r(r)} \times$$

$$(30) \quad \begin{vmatrix} \frac{p(n)}{(n+1)!} & \frac{p_{n-1}(n)}{1!} & \frac{p_{n-2}(n)}{2!} & \frac{p_{n-3}(n)}{3!} & \cdot & \cdot & \cdots & \cdots & \frac{p_0(n)}{n!} \\ \frac{p(n-1)}{n!} & \frac{p_{n-1}(n-1)}{0!} & \frac{p_{n-2}(n-1)}{1!} & \frac{p_{n-3}(n-1)}{2!} & \cdot & \cdot & \cdots & \cdots & \frac{p_0(n-1)}{(n-1)!} \\ \frac{p(n-2)}{(n-1)!} & 0 & \frac{p_{n-2}(n-2)}{0!} & \frac{p_{n-3}(n-2)}{1!} & \cdot & \cdot & \cdots & \cdots & \frac{p_0(n-2)}{(n-2)!} \\ \frac{p(n-3)}{(n-2)!} & 0 & 0 & \frac{p_{n-3}(n-3)}{0!} & \cdot & \cdot & \cdots & \cdots & \frac{p_0(n-3)}{(n-3)!} \\ \cdots & \cdots & \cdots & \cdots & \cdot & \cdot & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdot & \cdot & \cdots & \cdots & \cdots \\ \frac{p(2)}{3!} & 0 & 0 & 0 & \cdot & \cdot & \frac{p_2(2)}{0!} & \frac{p_1(2)}{1!} & \frac{p_0(2)}{2!} \\ \frac{p(1)}{2!} & 0 & 0 & 0 & \cdot & \cdot & \cdots & \frac{p_1(1)}{0!} & \frac{p_0(1)}{1!} \\ \frac{p(0)}{1!} & 0 & 0 & 0 & \cdot & \cdot & \cdots & \cdots & \frac{p_0(0)}{0!} \end{vmatrix}.$$

For example, from (30) for $n = 4$:

$$\frac{1}{5!} = -\frac{1}{25} \begin{vmatrix} \frac{p(4)}{5!} & \frac{p_3(4)}{1!} & \frac{p_2(4)}{2!} & \frac{p_1(4)}{3!} & \frac{p_0(4)}{4!} \\ \frac{p(3)}{4!} & \frac{p_3(3)}{0!} & \frac{p_2(3)}{1!} & \frac{p_1(3)}{2!} & \frac{p_0(3)}{3!} \\ \frac{p(2)}{3!} & 0 & \frac{p_2(2)}{0!} & \frac{p_1(2)}{1!} & \frac{p_0(2)}{2!} \\ \frac{p(1)}{2!} & 0 & 0 & \frac{p_1(1)}{0!} & \frac{p_0(1)}{1!} \\ p(0) & 0 & 0 & 0 & p_0(0) \end{vmatrix}$$

$$= -\frac{1}{25} \begin{vmatrix} \frac{5}{5!} & 0 & \frac{1}{2!} & 0 & 0 \\ \frac{4}{4!} & 5 & \frac{2}{1!} & 0 & 0 \\ \frac{3}{3!} & 0 & \frac{1}{0!} & \frac{-1}{1!} & 0 \\ \frac{2}{2!} & 0 & 0 & \frac{-1}{0!} & 0 \\ 1 & 0 & 0 & 0 & 1 \end{vmatrix} = \frac{1}{120}.$$

b). Jha [13] deduced the expression (26), which can be written in the form (27):

$$\frac{\sigma(n)}{n! n} = \sum_{j=0}^n \frac{p_{n-j}(n)}{j!} \frac{(-1)^{n-j}}{(n-j)! (n-j)}, \quad e_n = \frac{\sigma(n)}{n! n},$$

$$(31) \quad b_j^{(n)} = \frac{p_{n-j}(n)}{j!}, \quad c_{n-j} = \frac{(-1)^{n-j}}{(n-j)! (n-j)},$$

therefore:

$$\frac{(-1)^n}{n! n} = \frac{1}{\prod_{r=0}^n p_r(r)} \times$$

$$(32) \quad \begin{vmatrix} \frac{\sigma(n)}{(n)!n} & \frac{p_{n-1}(n)}{1!} & \frac{p_{n-2}(n)}{2!} & \frac{p_{n-3}(n)}{3!} & \cdot & \cdot & \cdots & \cdots & \frac{p_0(n)}{(n)!} \\ \frac{\sigma(n-1)}{(n-1)!(n-1)} & \frac{p_{n-1}(n-1)}{0!} & \frac{p_{n-2}(n-1)}{1!} & \frac{p_{n-3}(n-1)}{2!} & \cdot & \cdot & \cdots & \cdots & \frac{p_0(n-1)}{(n-1)!} \\ \frac{\sigma(n-2)}{(n-2)!(n-2)} & 0 & \frac{p_{n-2}(n-2)}{0!} & \frac{p_{n-3}(n-2)}{1!} & \cdot & \cdot & \cdots & \cdots & \frac{p_0(n-2)}{(n-2)!} \\ \frac{\sigma(n-3)}{(n-3)!(n-3)} & 0 & 0 & \frac{p_{n-3}(n-3)}{0!} & \cdot & \cdot & \cdots & \cdots & \frac{p_0(n-3)}{(n-3)!} \\ \cdots & \cdots & \cdots & \cdots & \cdot & \cdot & \cdots & \cdots & \frac{p_0(n-4)}{(n-4)!} \\ \cdots & \cdots & \cdots & \cdots & \cdot & \cdot & \cdots & \cdots & \cdots \\ \frac{\sigma(2)}{2!2} & 0 & 0 & 0 & \cdot & \cdot & \frac{p_2(2)}{0!} & \frac{p_1(2)}{1!} & \frac{p_0(2)}{2!} \\ \frac{\sigma(1)}{1!} & 0 & 0 & 0 & \cdot & \cdot & \cdots & \frac{p_1(1)}{0!} & \frac{p_0(1)}{1!} \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdots & \cdots & \frac{p_0(0)}{0!} \end{vmatrix}$$

From (32) for $n = 4$:

$$\frac{1}{4! \cdot 4} = -\frac{1}{25} \begin{vmatrix} \frac{7}{96} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{4}{18} & 5 & 2 & 0 & 0 \\ \frac{3}{4} & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = \frac{1}{96}.$$

c). Osler-Hassen-Chandrupatla [19] showed the property (22):

$$(33) \quad \sigma(n) = \sum_{k=0}^n k a_{n-k} p(k) = \sum_{j=0}^n a_j (n-j) p(n-j),$$

then (27), (28) and (33) imply the determinant:

$$(34) \quad n p(n) = \begin{vmatrix} \sigma(n) & a_1 & a_2 & a_3 & \cdots & \cdots & \cdots & a_n \\ \sigma(n-1) & 1 & a_1 & a_2 & \cdots & \cdots & \cdots & a_{n-1} \\ \sigma(n-2) & 0 & 1 & a_1 & a_2 & \cdots & \cdots & a_{n-2} \\ \cdots & \cdots & 0 & 1 & a_1 & \cdots & \cdots & a_{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma(1) & 0 & 0 & 0 & \cdots & \cdots & \cdots & a_1 \\ \sigma(0) & 0 & 0 & 0 & 0 & \cdots & \cdots & 1 \end{vmatrix}.$$

From (34) for $n = 4$:

$$4 p(4) = \begin{vmatrix} 7 & -1 & -1 & 0 & 0 \\ 4 & 1 & -1 & -1 & 0 \\ 3 & 0 & 1 & -1 & -1 \\ 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 20 \rightarrow p(4) = 5,$$

and for $n = 5$:

$$5 p(5) = \begin{vmatrix} 6 & -1 & -1 & 0 & 0 \\ 7 & 1 & -1 & -1 & 0 \\ 4 & 0 & 1 & -1 & -1 \\ 3 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 1 \end{vmatrix} = 35 \rightarrow p(5) = 7.$$

4. ON THE GANDHI'S RECURRENCE RELATION FOR COLOUR PARTITIONS

Gandhi [9, 17] showed the following recurrence relation for the colour partitions:

$$(35) \quad p_r(n) = -\frac{r}{n} \sum_{k=0}^n p_r(n-k) \sigma(k), \quad n \geq 1.$$

For $r = -1$ the relation (35) implies the known property (23) [19, 22, 1]:

$$(36) \quad np(n) = \sum_{k=0}^n p(n-k) \sigma(k) = \sum_{j=0}^n p(j) \sigma(n-j);$$

if $r = 1$, the Gandhi's expression (35) gives the Osler-Hassen-Chandrupatla's recurrence (24) [19], where it was applied the result (17). Jha [13] obtained the interesting identity (26) which can be considered as the inversion of (35).

Finally, with (24) and (35) it is possible to deduce the following property:

$$(37) \quad \frac{r}{r+1} p_{r+1}(n) = \frac{1}{n} \sum_{k=0}^n k a_{n-k} p_r(k), \quad r \geq 0, n \geq 1,$$

similar the relation:

$$(38) \quad p_{r+1}(n) = \sum_{k=0}^n a_{n-k} p_r(k).$$

Thus, we see that the Gandhi's formula involving the colour partitions and the divisor function implies interesting identities related with the partition function.

REFERENCES

- [1] C. Ballantine, M. Merca, *New convolutions for the number of divisors*, Journal of Number Theory 170(1) (2017), 17-34.
- [2] D. Birmajer, J. B. Gil, M. D. Weiner, *Linear recurrence sequences and their convolutions via Bell polynomials*, arXiv: 1405.7727v2 [math.CO] 26 Nov (2014).
- [3] D. Birmajer, J. B. Gil, M. D. Weiner, *Some convolution identities and an inverse relation involving partial Bell polynomials*, The Electr. J. of Combinatorics 19(4) (2012) P34, 1-4.
- [4] L. Comtet, *Advanced combinatorics*, D. Reidel, Dordrecht, Holland (1974).
- [5] Hei-Chi Chan, *An invitation to q -series. From Jacobi's triple product identity to Ramanujan's "most beautiful identity"*, World Scientific, Singapore (2011).
- [6] <https://en.wikipedia.org/wiki/Bell-polynomials>
- [7] <http://mathworld.wolfram.com/PartitionFunctionP.html>
- [8] A. D. Forbes, *Congruence properties of functions related to the partition function*, Pacific J. Math. 158(1) (1993), 145-156.
- [9] J.M. Gandhi, *Congruences for $p_r(n)$ and Ramanujan's t -functions*, Amer. Math. Monthly 70(3) (1963) 265-274.
- [10] H.W. Gould, *Combinatorial identities. Table I: Intermediate techniques for summing finite series*, Edited and compiled by J. Quaintance, May 3, (2010).
- [11] M.D. Hirschhorn, *Another short proof of Ramanujan's mod 5 partition congruences, and more*, Amer. Math. Monthly 106(6) (1999), 580-583.
- [12] W.P. Johnson, *An introduction to q -analysis*, Am. Math. Soc., Providence, Rhode Island, USA (2020).

- [13] S. Kumar Jha, *A combinatorial identity for the sum of divisors function involving $p_r(n)$* , *Integers* 20 (2020), A97.
- [14] S. Kumar Jha, *A formula for the r -coloured partition function in terms of the sum of divisors function and its inverse*, <https://arxiv.org/abs/2008.03106>, Preprint (2020).
- [15] S. Kumar Jha, *A formula for the number of partitions of n in terms of the partial Bell polynomials*, *The Ramanujan Journal*, Feb (2021), 1-4.
- [16] P. Lam-Estrada, J. López-Bonilla, *On the Jha's identity for the sum of inverses of odd divisors of a positive integer*, *Comput. Appl. Math. Soc.* 6(2) (2021), 27-29.
- [17] O. Lazarev, M. Mizuhara, B. Reid, *Some results in partitions, plane partitions, and multipartitions*, Summer 2010 REU Program in Maths. at Oregon State University, Aug 13 (2010).
- [18] J. Malenfant, *Finite, closed-form expressions for the partition function and for Euler, Bernoulli and Stirling numbers*, arXiv: 1103.1585v6 [math.NT] 24 May (2011).
- [19] T.J. Osler, A. Hassen, T. R. Chandrupatla, *Surprising connections between partitions and divisors*, *The College Maths. J.* 3(4) (2007), 278-287.
- [20] J. Quaintance, H. W. Gould, *Combinatorial identities for Stirling numbers*, World Scientific, Singapore (2016).
- [21] M. Shattuck, *Some combinatorial formulas for the partial r -Bell polynomials*, *Notes on Number Theory and Discrete Maths.* 23(1) (2017), 63-76.
- [22] R.P. Stanley, *Enumerative combinatorics. I*, Cambridge University Press (1999).
- [23] R. Vein, P. Dale, *Determinants and their applications in Mathematical Physics*, Springer-Verlag, New York (1999).

DEPARTAMENTO DE CIÊNCIAS EXATAS E TECNOLOGIAS, UNIVERSIDADE FEDERAL DO AMAPÁ, ROD. J. KUBITSCHKE, 68903-419, MACAPÁ, AP, BRAZIL,
E-mail address: `bulnes@unifap.br`

DEPARTMENT OF MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL 139 - 701, REPUBLIC OF KOREA,
E-mail address: `tkkim@kw.ac.kr`

ESIME-ZACATENCO, INSTITUTO POLITÉCNICO NACIONAL, EDIF. 4, 1ER. PISO, COL. LINDAVISTA, CP 07738, CDMX, MÉXICO,
E-mail address: `jlopezb@ipn.mx`

ESIME-ZACATENCO, INSTITUTO POLITÉCNICO NACIONAL, EDIF. 4, 1ER. PISO, COL. LINDAVISTA, CP 07738, CDMX, MÉXICO.
E-mail address: `svidalb@ipn.mx`