Article

Lorentz Transformation & the Intrinsic Geometry of a Time-like Curve in Minkowski Spacetime

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Abstract

We study the evolution of the Lorentz mapping, and of its associated unimodular complex matrix, between two Frenet-Serret's tetrads on an arbitrary time-like world line.

Keywords: Lorentz transformation, Rodrigues-Cartan's formula, Frenet-Serret's tetrad, special relativity, Dirac spinor.

1. Introduction

In this work we consider an arbitrary time-type trajectory in Minkowski space, whose intrinsic geometry is given by the well-known Frenet – Serret [FT] equations. parameterized with the proper time s of the particle under analysis. If in the initial event $s = 0$ we have a FT tetrad, it will evolve on the curve and rotate with respect to its original position, that is, both FT tetrads will be connected by a Lorentz transformation L, which in turn can be generated by a complex matrix B with a determinant equal to one. Here we obtain the exact equations that govern the evolution of L y B about the trajectory in terms of its intrinsic geometry [curvatures]; We point out that these equations admit immediate solution when the curve is a helix, thus obtaining the exact movement of a charged particle in a constant electromagnetic field. In our analysis, the expression of Olinde Rodrigues - Cartan is relevant, which establishes the connection between B y L. In addition, it is indicated that our study leads naturally to the matrix S that controls the transformation of the 4-spinor Dirac under Lorentz mappings. We exhibit a factorization of L and consider it important to investigate the possible physical meaning of this factorization [splitting].

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2. Frenet-Serret tetrads

The Olinde Rodrigues [1]-Cartan [2] expression [3-5]:

$$\begin{pmatrix} x^0 + x^3 & x^1 + i\,x^2 \\ x^1 - i\,x^2 & x^0 - x^3 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \tilde{x}^0 + \tilde{x}^3 & \tilde{x}^1 + i\,\tilde{x}^2 \\ \tilde{x}^1 - i\,\tilde{x}^2 & \tilde{x}^0 - \tilde{x}^3 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}, \tag{1}$$

where α, β, γ, δ are arbitrary complex numbers verifying the condition $\alpha\delta - \beta\gamma = 1$, implies six degrees of freedom for the Lorentz matrix $L = (L^\mu{}_\nu)$ between the frames of reference $(x^\nu) = (ct, x, y, z)$ and $(\tilde{x}^\mu)$:

$$x^\mu = L^\mu{}_\nu\, \tilde{x}^\nu\,. \tag{2}$$

From (1) and (2) we obtain the relations [5-13]:

$$\begin{aligned}
&L^0{}_0 = \tfrac{1}{2}\left(\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta}\right), L^1{}_0 = \tfrac{1}{2}\left(\bar{\alpha}\,\gamma + \bar{\beta}\delta\right) + cc, L^2{}_0 = -\tfrac{i}{2}\left(\alpha\bar{\gamma} - \bar{\beta}\delta\right) + cc\,,\\
&L^0{}_1 = \tfrac{1}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, L^1{}_1 = \tfrac{1}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, L^2{}_1 = -\tfrac{i}{2}\left(\alpha\bar{\delta} + \beta\bar{\gamma}\right) + cc,\\
&L^0{}_2 = -\tfrac{i}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, L^1{}_2 = -\tfrac{i}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, L^2{}_2 = \tfrac{1}{2}\left(\bar{\alpha}\delta - \bar{\beta}\gamma\right) + cc\,,\\
&L^0{}_3 = \tfrac{1}{2}\left(\alpha\bar{\alpha} - \beta\bar{\beta} + \gamma\bar{\gamma} - \delta\bar{\delta}\right), L^1{}_3 = \tfrac{1}{2}\left(\bar{\alpha}\gamma - \bar{\beta}\delta\right) + cc, L^2{}_3 = -\tfrac{i}{2}\left(\alpha\bar{\gamma} + \bar{\beta}\delta\right) + cc\,,\\
&L^3{}_0 = \tfrac{1}{2}\left(\alpha\bar{\alpha} + \beta\bar{\beta} - \gamma\bar{\gamma} - \delta\bar{\delta}\right), L^3{}_1 = \tfrac{1}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc, L^3{}_2 = -\tfrac{i}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc\,,\\
&L^3{}_3 = \tfrac{1}{2}\left(\alpha\bar{\alpha} - \beta\bar{\beta} - \gamma\bar{\gamma} + \delta\bar{\delta}\right), \alpha\delta - \beta\gamma = 1,
\end{aligned} \tag{3}$$

where *cc* means the complex conjugate of all the previous terms.

On the other hand, in Minkowski spacetime two real orthonormal tetrads are connected by a Lorentz transformation [14]:

$$e^{(a)}{}_\mu = L^a{}_b\, \tilde{e}^{(b)}{}_\mu\,, \tag{4}$$

similar to (2), then (1) is applicable with $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $B^\dagger = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$:

$$\begin{pmatrix} e^{(0)}{}_{\mu}+e^{(3)}{}_{\mu} & e^{(1)}{}_{\mu}+i\,e^{(2)}{}_{\mu} \\ e^{(1)}{}_{\mu}-i\,e^{(2)}{}_{\mu} & e^{(0)}{}_{\mu}-e^{(3)}{}_{\mu} \end{pmatrix} = B\begin{pmatrix} \tilde{e}^{(0)}{}_{\mu}+\tilde{e}^{(3)}{}_{\mu} & \tilde{e}^{(1)}{}_{\mu}+i\,\tilde{e}^{(2)}{}_{\mu} \\ \tilde{e}^{(1)}{}_{\mu}-i\,\tilde{e}^{(2)}{}_{\mu} & \tilde{e}^{(0)}{}_{\mu}-\tilde{e}^{(3)}{}_{\mu} \end{pmatrix}B^{\dagger}, \tag{5}$$

and we can make the identifications:

$$e^{(0)}{}_{\mu}=t_{\mu}\,,e^{(j)}{}_{\mu}=-\,n_{(j)\mu}\,,j=1,2,3, \tag{6}$$

where $(t^{\mu},\, n_{(1)}{}^{\mu}\,,\, n_{(2)}{}^{\mu}\,,\, n_{(3)}{}^{\mu}\,)$ is a Frenet-Serret's tetrad verifying the known relations [15-21]:

$$\begin{gathered} \frac{d}{ds}t^{\mu}=k_1 n_{(1)}{}^{\mu}, \qquad \frac{d}{ds}n_{(1)}{}^{\mu}=k_1 t^{\mu}+k_2 n_{(2)}{}^{\mu}, \\ \frac{d}{ds}n_{(2)}{}^{\mu}=-k_2 n_{(1)}{}^{\mu}+k_3 n_{(3)}{}^{\mu}, \qquad \frac{d}{ds}n_{(3)}{}^{\mu}=-k_3 n_{(2)}{}^{\mu}, \end{gathered} \tag{7}$$

connecting the tangent and normal vectors with the principal curvatures on a time-like trajectory in Minkowski geometry. Then (5) acquires the form:

$$\begin{pmatrix} t^{\mu}-n_{(3)}{}^{\mu} & -n_{(1)}{}^{\mu}-i\,n_{(2)}{}^{\mu} \\ -n_{(1)}{}^{\mu}+i\,n_{(2)}{}^{\mu} & t^{\mu}+n_{(3)}{}^{\mu} \end{pmatrix} = B\begin{pmatrix} \tilde{t}^{\mu}-\tilde{n}_{(3)}{}^{\mu} & -\tilde{n}_{(1)}{}^{\mu}-i\,\tilde{n}_{(2)}{}^{\mu} \\ -\tilde{n}_{(1)}{}^{\mu}+i\,\tilde{n}_{(2)}{}^{\mu} & \tilde{t}^{\mu}+\tilde{n}_{(3)}{}^{\mu} \end{pmatrix}B^{\dagger}\,, \tag{8}$$

being $\tilde{t}^{\mu}\,,\tilde{n}_{(j)}{}^{\mu}$ the tetrad for $s=0$.

The expression (8) must be compatible with (7), therefore we propose the following evolution law for B on the world line such that (8) generates the FT's formulae:

$$\frac{d}{ds}\,B=\Phi\,B\,,\Phi=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{9}$$

then from (8) and (9):

$$\frac{d}{ds}\begin{pmatrix} t^{\mu}-n_{(3)}{}^{\mu} & -n_{(1)}{}^{\mu}-i\,n_{(2)}{}^{\mu} \\ -n_{(1)}{}^{\mu}+i\,n_{(2)}{}^{\mu} & t^{\mu}+n_{(3)}{}^{\mu} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} t^{\mu}-n_{(3)}{}^{\mu} & -n_{(1)}{}^{\mu}-i\,n_{(2)}{}^{\mu} \\ -n_{(1)}{}^{\mu}+i\,n_{(2)}{}^{\mu} & t^{\mu}+n_{(3)}{}^{\mu} \end{pmatrix}+$$

$$+\begin{pmatrix} t^{\mu}-n_{(3)}{}^{\mu} & -n_{(1)}{}^{\mu}-i\,n_{(2)}{}^{\mu} \\ -n_{(1)}{}^{\mu}+i\,n_{(2)}{}^{\mu} & t^{\mu}+n_{(3)}{}^{\mu} \end{pmatrix}\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix},$$

whose comparison with (7) implies the values:

$$a = -\, e = -\tfrac{i}{2}\, k_2\,, b = c = -\tfrac{1}{2}\,(k_1 + i\, k_3), \tag{10}$$

hence (9) takes the structure [22]:

$$\frac{d}{ds}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = -\frac{1}{2}\begin{pmatrix} i\,k_2 & k_1 + i\,k_3\\ k_1 + i\,k_3 & -i\,k_2\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \tag{11}$$

which allows determine $B(s)$ with the initial condition $B(0) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. The property (11) indicates that the intrinsic geometry of the world line governs the evolution of $\alpha, \beta, \gamma, \delta$ and this in turn, via (3), gives the corresponding Lorentz transformation in each point of the time-like curve.

Remark 1.- If we know B, then with (3) we construct the Lorentz matrix; the inverse problem is to obtain B if we have L, and the answer is [22-25]:

$$B = (B^A{}_C) = \pm\, \frac{1}{K}\, L^\mu{}_\nu\, \sigma_\mu{}^{A\dot{E}}\, \sigma^\nu{}_{C\dot{E}}, K \equiv \sqrt{\det(L^\tau{}_\lambda\, \sigma_\tau{}^{Q\dot{F}} \sigma^\lambda{}_{R\dot{F}})}\,, \tag{12}$$

in terms of the Infeld-van der Waerden symbols [9, 26-28].

The helices are special curves because their curvatures are constants [21], then it is immediate the integration of (11):

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}(s) = \exp\left[-\frac{1}{2}\begin{pmatrix} i\,k_2 & k_1 + i\,k_3\\ k_1 + i\,k_3 & -i\,k_2\end{pmatrix} s\right]\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{13}$$

and in [19, 29, 30] is given the exact formula for the exponential function of an arbitrary 2 x 2 matrix. Synge [15, 31] proved that the trajectory of a classical charged particle under a uniform electromagnetic field is a helix in the spacetime, which allows integrate the corresponding Lorentz equation [15, 19, 22, 32-46].

Remark 2.- We note that the relations (2) imply the interesting factorization for the Lorentz matrix [13, 47, 48]:

$$L = L_1\, L_2\,, L_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha & \beta & \gamma & \delta\\ \gamma & \delta & \alpha & \beta\\ i\gamma & i\delta & -i\alpha & -i\beta\\ \alpha & \beta & -\gamma & -\delta\end{pmatrix}, L_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}\bar{\alpha} & \bar{\beta} & i\bar{\beta} & \bar{\alpha}\\ \bar{\beta} & \bar{\alpha} & -i\bar{\alpha} & -\bar{\beta}\\ \bar{\gamma} & \bar{\delta} & i\bar{\delta} & \bar{\gamma}\\ \bar{\delta} & \bar{\gamma} & -i\bar{\gamma} & -\bar{\delta}\end{pmatrix}, \tag{14}$$

such that $det\, L_1 = -\, det\, L_2 = i$, whose possible physical meaning is an open problem.

Remark 3.- If we apply $\frac{d}{ds}$ to (4) and employ (6) and (7) it is immediate the evolution of the Lorentz matrix in terms of the intrinsic geometry of the time-like trajectory:

$$\frac{d}{ds} L^a{}_b = L^a{}_r F^r{}_b \, , (F^r{}_b) = \begin{pmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & -k_2 & 0 \\ 0 & k_2 & 0 & -k_3 \\ 0 & 0 & k_3 & 0 \end{pmatrix}, \tag{15}$$

as an alternative to (11).

Remark 4.- The relation (11) implies the following expressions:

$$\frac{d\alpha}{ds} = -\frac{1}{2} \left[i\, k_2\, \alpha + (k_1 + i\, k_3)\, \gamma \right], \frac{d\beta}{ds} = -\frac{1}{2} \left[i\, k_2\, \beta + (k_1 + i\, k_3)\, \delta \right],$$
$$\frac{d\gamma}{ds} = -\frac{1}{2} \left[-i\, k_2\, \gamma + (k_1 + i\, k_3)\, \alpha \right], \frac{d\delta}{ds} = -\frac{1}{2} \left[-i\, k_2\, \delta + (k_1 + i\, k_3)\, \beta \right], \tag{16}$$

which, in natural manner, allow to construct the evolution of an interesting matrix [49]:

$$\frac{d}{ds} S = \frac{i}{2} \begin{pmatrix} k_2 & k_3 & 0 & ik_1 \\ k_3 & -k_2 & ik_1 & 0 \\ 0 & ik_1 & k_2 & k_3 \\ ik_1 & 0 & k_3 & -k_2 \end{pmatrix} S, \qquad S = \begin{pmatrix} A & E \\ E & A \end{pmatrix}, \tag{17}$$

where:

$$A = \frac{1}{2} \begin{pmatrix} \bar{\alpha} + \delta & \bar{\beta} - \gamma \\ \bar{\gamma} - \beta & \alpha + \bar{\delta} \end{pmatrix}, E = \frac{1}{2} \begin{pmatrix} \bar{\alpha} - \delta & \bar{\beta} + \gamma \\ \beta + \bar{\gamma} & \bar{\delta} - \alpha \end{pmatrix}, \tag{18}$$

and the surprise is that S gives the transformation of the Dirac 4-spinor ψ under Lorentz mappings [50, 51], that is, $\tilde{\psi} = S\, \psi$.

3. Conclusions

With the Frenet-Serret´s tetrad we can construct a null tetrad of Newman-Penrose [NP] [9, 26], besides, B and $B^\dagger$ are associated with 2-spinors, then we consider that our analysis admits the spinorial and NP formalisms [36]. The study here realized is useful in the description of the motion of classical particles charged [15, 19, 22, 31-46]. The integration of (11), (15) and (17) for the case of constant curvatures indicates the importance of efficient methods to determine the exponential function of a matrix [19, 29, 30, 52].

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