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Certain integrals involving legendre polynomials

Abstract

Here we exhibit alternative proofs of the identities given by Persson-Strang and (Huat-Chan)- -Wan-Zudilin for the Legendre polynomials. Besides, we show the connection between the Lanczos derivative and these polynomials via the Rangarajan-Purushothaman's formula.

Keywords: (Huat-Chan)-Wan-Zudilin's property, Legendre polynomials, Persson-Strang's identity, Rangarajan-Purushothaman's expression, Lanczos derivative

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Introduction

The Legendre's polynomials[1] $P_n(x), -1 \le x \le 1$, can be defined via the following recurrence relation:[2–4]

$$(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}, P_0 = 1, P_1 = x, n = 1,2..., \quad (1)$$

hence:

$$P_2 = \frac{1}{2}\left(3x^2-1\right), P_3 = \frac{1}{2}\left(5x^3-3x\right), P_4 = \frac{1}{8}\left(35x^4-30x^2+3\right), ... \quad (2)$$

These polynomials also are determined univocally through the conditions;[5,6]

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0, m \ne n, P_n(1) = 1, \forall n, \quad (3)$$

therefore:

$$\int_{-1}^{1} x^m P_n(x)\,dx = 0, m < n, \quad (4)$$

and the Laplace's integral formula[3–5,7] gives an alternative way to generate the expressions (2):

$$P_n(x) = \frac{1}{2^n}\int_{-\pi}^{\pi}\left(x+\sqrt{x^2-1}\cos\beta\right)^n d\beta, n = 0,1,2... \quad (5)$$

or equivalently:

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k\binom{n}{k}\binom{2n-2k}{n}x^{n-2k}. \quad (6)$$

Persson-Strang & (Huat-Chan)-Wan-Zudilin identities

Here we have interest in the value of the following integral:

$$Q(m) \equiv \int_{-1}^{1}\frac{1}{x}P_{2m+1}(x)\,dx, m = 01,2,.. \quad (7)$$

then from (6) with $n = 2m+1$:

$$Q(m) = \frac{1}{2^n}\int_{-1}^{1}\sum_{k=0}^{m}(-1)^k\binom{n}{k}\binom{2n-2k}{n}x^{2m-2k}dk = \frac{1}{4^m}A(m), \quad (8)$$

where $A(m)$ can be calculated via the method of Petkovsek-Wilf-Zeilberger,[8–18] in fact:

$$A(m) \equiv \sum_{k=0}^{m}\frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!(n-2k)} = \frac{(2n)!}{n(n!)^2}\sum_{k=0}^{\infty}t_k, t_k = \frac{(-1)^k n(n!)^2(2n-2k)!}{(2n)!k!(n-k)!(n-2k)!(n-2k)}, \quad (9)$$

Therefore $\frac{t_{k+1}}{t_k} = \frac{\left(k-m-\frac{1}{2}\right)^2(k-m)}{\left(k-m+\frac{1}{2}\right)\left(k-2m-\frac{1}{2}\right)(k+1)}$, hence:

$$A(m) \equiv \frac{(2n)!}{n(n!)^2}3F_2\left(-m,-m-\frac{1}{2},-m-\frac{1}{2};-m+\frac{1}{2},-2m-\frac{1}{2};1\right) = \frac{(-1)^m 4^n (m!)^2}{2(n!)}, n = 2m+1, \quad (10)$$

where it was applied the following value of the hypergeometric function in (10):

$$3F_2(\;) = \frac{(-16)^m n!(m!)^2}{(4m+1)!}. \quad (11)$$

then (8) and (10) imply the result:

$$Q(m) = \frac{2(-4)^m(m!)^2}{(2m+1)!}. \quad (12)$$

On the other hand, from (6) for $n = 2m+1$:

$$\left[\frac{P_{2m+1}(x)}{x}\right]^2 = \frac{1}{2^n}\sum_{k=0}^{m}(-1)^k\binom{n}{k}\binom{2n-2k}{n}x^{2m-2k}\frac{P_{2m+1}(x)}{x},$$

where we can integrate in the interval $[-1,1]$ and apply the properties (4) and (12) to obtain the relation:

$$\int_{-1}^{1}\left[\frac{P_{2m+1}(x)}{x}\right]^2 dx = \frac{(-1)^m}{2^n}\binom{n}{k}\binom{2n-2k}{n}Q(m) = 2, m = 0,1,2,... \quad (13)$$

deduced by Persson-Strang;[19] Amdeberhan et al.[20] generalized the identity (13) in the form:

$$\int_{-1}^{1}\left[\frac{P_l(x)-P_l(0)}{x}\right]^2 dx = 2[1-\beta^2(l)], l = 0,1,2,... \quad (14)$$

such that:

$$\beta(l) = \left\{2-1\binom{l}{1/2}\begin{matrix}0, if\ is\ odd\\ , if\ l\ is\ even\end{matrix}\right. . \quad (15)$$

Remark. - In (6) we may use $x = \frac{b}{\sqrt{b^2-4c}}$ to obtain:

$$P_n\left(\frac{b}{\sqrt{b^2-4c}}\right) = \frac{1}{\left(b^2-4c\right)^{n/2}}\sum_{j=0}^{\lfloor n/2 \rfloor}\frac{b^{n-2j}(-4c)^j}{2^n j!}R(n) \quad (16)$$

where:

$$R(n) \equiv \sum_{k=j}^{\lfloor n/2 \rfloor}\frac{(-1)^k(2n-2k)!}{(2n-2k)!(k-i)!(n-k)!} = \frac{(-1)^j 2^{n-2j}n!}{j!(n-2j)!}, 0 \le j \le \lfloor n/2 \rfloor \quad (17)$$

thus (16) and (17) imply the interesting identity of (Huat-Chan)-Wan-Zudilin:[21,22]

$$\left(b^2-4c\right)^{n/2}P_n\left(\frac{b}{\sqrt{b^2-4c}}\right) = \sum_{j=0}^{\lfloor n/2 \rfloor}\binom{n}{2j}\binom{2j}{j}b^{n-2j}c^j \quad (18)$$

We may indicate two useful relations:[23,24]

$$[P_n(x)]^2 = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{n}\binom{2k}{k}\left(-\frac{1-x^2}{4}\right)^k, n=0,1,2,... \quad (19)$$

$$\int_{-1}^{1} x^m P_n(x)\,dx = \frac{2^{n+1}}{m+1}\cdot\frac{\binom{\frac{m+n}{2}}{n}}{\binom{m+n+1}{n}}, m-n=0,2,4,... \quad (20)$$

We emphasize the importance of the method of Petkovsek-Wilf-Zeilberger to obtain (10) and (17).

Lanczos generalized derivative

Rangarajan-Purushothaman[25,26] obtained the following generalization of the Lanczos derivative:[27,28]

$$f^{(m)}(x) = lim_{\varepsilon\to 0}\frac{(2m+1)!!}{2\varepsilon^{m+1}}\int_{-\varepsilon}^{\varepsilon} P_m\left(\frac{t}{\varepsilon}\right) f(x+t)\,dt, m=1,2,... \quad (21)$$

involving the Legendre polynomials.

If $f(x)=1,$ then (21) implies the property:

$$\int_{-1}^{1} P_n(u)\,du = 0, n=2,4,6,... \quad (22)$$

From (21) for $f(x)=x^N$:

$$\int_{-1}^{1} P_n(u)u^k\,du = 0,\ k<n, \quad (23)$$

$$\int_0^1 P_n(u)u^n\,du = \frac{n!}{(2n+1)!!} = \frac{2^n (n!)^2}{(2n+1)!}, n=0,2,... \quad (24)$$

On the other hand, we know the relations:

$$\int_0^1 P_{2l}(u)u^m\,du = \frac{(-1)^l\,\Gamma\left(l-\frac{m}{2}\right)\Gamma\left(\frac{m+1}{2}\right)}{2\Gamma\left(-\frac{m}{2}\right)\Gamma\left(l+\frac{m+3}{2}\right)}, m>-1, \quad (25)$$

$$\int_0^1 P_{2l+1}(u)u^m\,du = \frac{(-1)^l\,\Gamma\left(l+\frac{1-m}{2}\right)\Gamma\left(1+\frac{m}{2}\right)}{2\Gamma\left(1+2+\frac{m}{2}\right)\Gamma\left(\frac{1-m}{2}\right)}, m>-2, \quad (26)$$

thus (24) can be deduced from (25) and (26) for $m=n=2l$ and $m=n=2l+1,$ respectively.

We have the following Schmied's formula (2005):[29]

$$u^m = \sum_{l=m,m-2,...}\frac{m!(2l+1)}{2^{\frac{m-1}{2}}\left(\frac{m-1}{2}\right)!(m+l+1)!!}P_l(u), \quad (27)$$

which gives (20), and for $m=n$ implies (24).

The Legendre polynomials can be written in terms of the Gauss hypergeometric function:

$$P_n(0) = \frac{(2n-1)!!}{n!}\sum_{k=0}^{n}\binom{n}{k}\,{}_2F_1(k-n,-n;-2n;2)x^k, \quad (28)$$

and we know the result:

$${}_2F_1(-n,-n;-2n;2) = \begin{cases} 0, n=1,3,5,... \\ \frac{(1-)^{\frac{n}{2}} n!(n-1)!!}{n!!(2n-1)!!}, n=2,4,6,... \end{cases}, \quad (29)$$

then from (28) and (29):

$$P_n(0) = {}_2F_1\left(-n,n+1;1;\frac{1}{2}\right) = \begin{cases} 0, n=1,3,5,... \\ \frac{(1-)^{\frac{n}{2}} n!(n-1)!!}{n!!}, n=2,4,6,... \end{cases}. \quad (30)$$

Finally, the expression:

$$P_n(x) \equiv \frac{1}{2^n}(-1)^k (1-x)^k (1+x)^{n-k}\binom{n}{k}^2, \quad (31)$$

and (30) imply the relation:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = \begin{cases} 0, n=1,3,5,... \\ \frac{(1-)^{\frac{n}{2}} 2^n (n-1)!!}{n!!}, n=2,4,6,... \end{cases}. \quad (32)$$

Thus, we see that the Rangarajan-Purushothaman's formula for the Lanczos derivative allows deduce some properties of Legendre polynomials, and it represents differentiation by integration. The $P_n(x)$ are orthogonal polynomials, hence Diekema-Koornwinder[30] consider that the name "orthogonal derivative" is adequate for (21).

Remark. - From (3) we have the property $P_n(1)=1\forall n,$ then (6) for $x=1$ gives the identity:

$$2n = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k\binom{n}{k}\binom{2n-2k}{n}; \quad (33)$$

on the other hand, we know the relation:[31]

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{z+ky}{n} = (-y)^n, y\neq 0, \quad (34)$$

which for $y=-2$ and $z=2n$ is equivalent to (33) because $\binom{2n-2k}{n}=0$ for $k>\lfloor\frac{n}{2}\rfloor$.

Finally, we consider that the publications[32–37] have useful relationship with the study realized in the present paper.

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Conflicts of interest

The author declares there is no conflict of interest.

References

1. AM Legendre. Mémoires de mathématique et de physique : prés. à l´Académie Royale des Sciences, par divers savans, et lûs dans ses assemblées. *Universitätsbibliothek Johann Christian Senckenberg*. 1785;10:411–435.
2. Lanczos C. *Legendre versus Chebyshev polynomials,* in "Topics in Numerical Analysis". In: JJH Miller editor. (Proc. Roy. Irish Acad. Conf. on Numerical Analysis, Aug. 14-18, 1972), Academic Press, London. 1973. p. 191–201.
3. Chihara TS. *An introduction to orthogonal polynomials,* Gordon & Breach, New York. 1978. p. 1–5.
4. Oldham KB, Spanier J. *An atlas of functions*. Hemisphere Pub. Co., London. 1987.
5. Sommerfeld A. *Partial differential equations in PhysicS*. Academic Press, New York. 1964.
6. Broman A. *Introduction to partial differential equations: From Fourier series to boundary-value problems*. Dover, New York. 1989.
7. López-Bonilla J, López-Vázquez R, Torres-Silva H. On the Legendre polynomials. *Prespacetime Journal*. 2015;6(8):735–739.
8. Petkovsek M, Wilf HS, ZeilbergerD. *A = B, symbolic summation algorithms*. In: AK Peters editor, Wellesley, Mass. USA. 1996.
9. Koepf W. *Hypergeometric summation. An algorithmic approach to summation and special function identities*. Vieweg, Braunschweig/ Wiesbaden. 1998.

10. Koepf W. Orthogonal polynomials and recurrence equations, operator equations and factorization. *Electronic Transactions on Numerical Analysis*. 2007;27:113–123.
11. Hannah JP. *Identities for the gamma and hypergeometric functions: an overview from Euler to the present.* Master of Science Thesis, University of the Witwatersrand, Johannesburg, South Africa. 2013.
12. Guerrero-Moreno I, López-Bonilla J. Combinatorial identities from the Lanczos approximation for gamma function. *Comput Appl Math Sci*. 2016;1(2):23–24.
13. López-Bonilla J, López-Vázquez R, Vidal-Beltrán S. Hypergeometric approach to the Munarini and Ljunggren binomial identities. *Comput Appl Math Sci*. 2018;3(1):4–6.
14. Barrera-Figueroa V, Guerrero-Moreno I, López-Bonilla J, et al. Some applications of hypergeometric functions. *Comput Appl Math Sci*. 2018;3(2):23–25.
15. Léon-Vega CG, López-Bonilla J, Vidal-Beltrán S. On a combinatorial identity of Cheon-Seol-Elmikkawy. *Comput Appl Math Sci*. 2018;3(2):31–32.
16. López-Bonilla L, Miranda-Sánchez I. Hypergeometric version of a combinatorial identity. *Comput Appl Math Sci*. 2020;5(1):6–7.
17. López-Bonilla J, Morales-García M. Hypergeometric version of Engbers-Stocker's combinatorial identity. *Studies in Nonlinear Sci*. 2021;6(3):41–42.
18. López-Bonilla J, Ovando G. On q-Hypergeometric series. *Studies in Nonlinear Sci*. 2021;6(4):56–58.
19. Persson PE, Strang G. Smoothing by Savitzky-Golay and Legendre filters, in "Mathematical systems theory of biology, communications, computations, and finance". *IMA*. 2003;134:301–316.
20. Amdeberhan T, Duncan A, Moll VH, et al. Filter integrals for orthogonal polynomials. *Hardy- Ramanujan Journal*. 2021;44:116–135.
21. Huat-Chan H, Wan J, Zudilin W. Legendre polynomials and Ramanujan-type series for 1/π. *Israel J of Maths*. 2013;194(1):183–207.
22. López-Bonilla L, López-Vázquez R, Vidal-Beltrán S. On an identity involving Legendre polynomials. *Studies in Nonlinear Sci*. 2019;4(2):10–11.
23. Zudilin W. *A generating function of the squares of Legendre polynomials*. 2012. p. 7.
24. Guerrero-Moreno I, López-Bonilla J, Zúñiga-Segundo A. Binomial identities via Legendre polynomials. *Open J Appl Theor Maths*. 2017;3(3):1–3.
25. Rangarajan SK, Purushothaman SP. Lanczos generalized derivative for higher orders. *J Comp Appl Maths*. 2005;177(2):461–465.
26. López-Bonilla J, López-Vázquez R, Vidal-Beltrán S. Orthogonal derivative for higher orders. *Comput Appl Math Sci*. 2018;3(1):7–8.
27. Lanczos C. *Applied analysis*. Dover, New York. 1988.
28. Hernández-Galeana A, Laurian Ioan P, López-Bonilla J, et al. On the Cioranescu-(Haslam-Jones)-Lanczos generalized derivative. *Global J Adv Res Class Mat Geom*. 2014;3(1):44–49.
29. Diekema E, Koornwinder T. Differentiation by integration using orthogonal polynomials, a survey. *J of Approximation Theory*. 2012;164:637667.
30. Quaintance J, Gould HW. Combinatorial identities for Stirling numbers. World Scientific, Singapore. 2016.
31. Mishra VN, Mishra LN. Trigonometric approximation of signals (functions) in Lp -norm. *Int J of Contemporary Math Sci.* 2012;7(19):909–918.
32. Mishra VN, Khatri K, Mishra LN. Using linear operators to approximate signals of Lip $(\alpha, p), (p \geq 1)$ – class. *Filomat.* 2013;27(2):353–363.
33. Mishra VN, Khatri K, Mishra LN, et al. Trigonometric approximation of periodic signals belonging to generalized weighted Lipschitz W(Lr, $\xi(t)), (r \geq 1)$ – class by Nörlund – Euler (N, p_n) (E, q) operator of conjugate series of its Fourier series. *J of Classical Anal*. 2014;5(2):91–105.
34. Mishra LN, Mishra VN, Khatri K, et al. On the trigonometric approximation of signals belonging to generalized weighted Lipschitz W(L^r, $\xi(t), (r \geq 1) -$ class by matrix $(C^1 \cdot N_p)$ operator of conjugate series of its Fourier series. *Appl Maths and Comp*. 2014;237:252–263.
35. Sahani SK, Mishra VN, Pahari NP. Some problems on approximations of functions (signals) in matrix summability of Legendre series. *Nepal J of Math Sci*. 2021;2(1):43–50.
36. Mishra LN, Raiz M, Rathour L, et al. Tauberian theorems for weighted means of double sequences in intuitionistic fuzzy normed spaces. *Yugoslav J of Operations Res*. 2022;32(3):277–388.

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